MECHANICS OF IMMISCIBLE FLUIDS IN POROUS MEDIA

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by Arthur T. Corey

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PREFACE TO THIRD EDITION

The primary objective of this text is the presentation of basic principles of the mechanics of two-phase fluid systems in soils and porous rocks. It is not designed as a reference book or as a thorough review of the literature. The scope of the material presented is limited to what can be covered in a one-semester course.

Since publication of the first edition in 1977, pollution of groundwater aquifers by petroleum fluids and other non-aqueous liquids, such as chlorinated hydrocarbons, has become a major concern of investigators both nationally and internationally. Material presented in this text provides necessary background for students interested in the analysis of such problems.

Changes appearing in the third edition are made with the objective of providing a clearer presentation of basic principles. Major changes appear in Chapters 4, 5, and 6, which have been rewritten to provide new material and a more rigorous treatment of the theory for flow of two immiscible fluids.

An effort is made to make the subject of similarity understandable to students, a majority of whom have previously been exposed only briefly to dimensional analysis. The analysis presented in Chapter VI derives criteria of similarity from equations governing simultaneous flow of two immiscible fluids. This procedure identifies similarity requirements more directly and in a more useful form than is possible from a dimensional analysis considering only the variables involved in two-phase flow.

Thanks are due Professor Jacob H. Dane of Auburn University for reviewing the revised manuscript and providing significant suggestions that have been included in the third edition. Thanks are also due Professor David B. McWhorter who has reviewed the manuscript and contributed the material in Section 5.3 dealing with linear imbibition with nonwetting phase resistance.

August 1994
Fort Collins, Colorado
U.S.A.

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Professor Emeritus
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PREFACE TO FIRST EDITION

This text is concerned primarily with the mechanics of two-phase fluid systems in soils and porous rocks. Applications relating to infiltration, subsurface drainage and production of oil and gas from petroleum reservoirs are included. Each of these applications involves the replacement of one fluid by another, for example, water by air or the reverse and, consequently, are beyond the scope of most texts dealing with groundwater hydrology. Replacement processes, however, are relevant to the overall analysis of groundwater hydrology because they are involved in the interchange between groundwater and water in the atmosphere through the surface mantle. Therefore, the material presented should be of interest to students of hydrology as well as to students of soil science and petroleum reservoir engineering.

The fields of possible applications are extremely diverse. Although the primary sources for material presented in the text are journals of the petroleum industry and soil science, contributions have come from the literature of chemical, agricultural and civil engineers and also from biologists, physicists, applied mathematicians and people working in various industries. An important objective of this text is to prepare students to read literature dealing with replacement processes in porous media regardless of the origin of the literature.

It has often happened in the past that scientists in particular fields have ignored the literature generated by others, perhaps because of the inconvenience of digesting unfamiliar notations and somewhat different viewpoints or because of a lack of awareness of the existence of literature on the subject in unfamiliar journals. This text attempts to provide an insight into the relationship among the various viewpoints and an awareness of the extensive literature and broad application for the material presented.

Although reference is made to material from diverse sources, the text does not provide an exhaustive literature review. In the first three chapters particularly, where the emphasis is upon the presentation of basic concepts, little effort is made to present the historical development of the concepts. The motivation for this approach is the observation that students are often distracted by excessive references. In the last three chapters, where concepts are presented which are still undergoing research and change, the sources for the material are given.

The text does not present the broad subject of transport in porous media in all its aspects. The phenomena considered are those that can be analyzed from the viewpoint of traditional fluid mechanics, that is, by assuming that each of the two fluid phases constitutes a continuum. The assumptions are made that both fluid phases are Newtonian viscous fluids, that they undergo negligible compression or expansion, and that the processes considered are not affected significantly by temperature variations, chemical reactions, or changes of phase. Consideration is not given to processes taking place on a molecular scale. For example, diffusion involving molecular mixing in response to a concentration gradient is not treated. The analysis is based upon the statics and dynamics of fluid particles as the term “particle” is used in fluid mechanics. Consideration is given also to properties pertaining to macroscopic elements of the porous matrix which include solids as well as fluids. In this sense, the subject is treated from both a “microscopic” and a “macroscopic” point of view.

Because of the restrictions upon the material presented, the text is expected to be useful mainly for students in programs that provide other courses relevant to transport in porous media including groundwater hydrology, soil physics and courses treating potential flow theory, the solution of boundary value problems, etc. For students with an appropriate background, it is expected that the material presented can be covered in a one-semester course.

The text is designed primarily for students having a background in elementary fluid mechanics and applied mathematics. Equations are written using the Einstein summation convention which is explained where it is introduced.

Much of the material used in the text is taken from courses taught by the author at Colorado State University during the years 1956 through 1976. The research and ideas of many former graduate students at this University are incorporated in the material presented. The encouragement of the administration and colleagues at the University in the preparation of the text is appreciated. Thanks are due Drs. S. C. Jones of the Marathon Oil Company and E. E. Miller of the University of Wisconsin whose suggestions are incorporated in Chapters 5 and 6 respectively, and P. J. Shuler of the University of Colorado who has reviewed the entire text and made many useful suggestions.

Thanks are also due the author's wife, Vera, for final editing of the entire text and Mrs. Betty D. Hutcheson for typing the manuscript.

January, 1977

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LIST OF SYMBOLS

The symbols listed are those that appear in more than one chapter or in several sections of the text. The section and page listed is where the meaning of the symbol is described.

Symbol		Section	Page
$D(\boldsymbol{\theta})$	Coefficient used in the diffusion form of the Richards equation	3.9	111
e	a subscript meaning effective	2.3.3	42
e	a unit vector		
F_w	fractional flow function	5.4.5	205
F	force vector		
f	a function of permeability and viscosity	5.3	186
	ratios for the wetting fluid		
F.C.	field capacity	2.3.4	42
g	magnitude of force per unit mass due to gravity--a scalar		
g	force due to gravity -- a vector		
H	piezometric head -- $p/\gamma + h$	3.2.3	80
h	elevation above a datum		
h_c	capillary pressure head -- p_c / γ		
h_d	elevation in a soil-water system at which $p_c = p_d$.		
i, j, k	subscripts referring to orthogonal directions		
i, j, k	unit vectors in orthogonal directions		
K	hydraulic conductivity	3.8	99

k	permeability (intrinsic permeability)	3.7	97
k_r	relative permeability	3.7	97
k_s	shape factor	3.3.4	86
m	subscript indicating a critical or maximum value		
nw	subscript referring to nonwetting fluid		
p	fluid pressure	2.1.1	29
p_c	capillary pressure	1.5	13
p_d	displacement pressure	2.3.2	40
p_e	entry pressure	2.3.2	40
p^*	piezometric pressure	3.2.3	80
q	volume flux	3.7	96
q_t	total flux = $q_w + q_{nw}$	5.3	185
R	hydraulic radius	3.4	87
r	(1) radius, or		
	(2) subscript indicating a residual saturation	2.3.3	41
	(3) subscript meaning relative	3.7	97
S	saturation	1.4.4	11
S_e	effective saturation	2.3.3	42
s	specific surface	1.3.2	5
$\mathbf{s}$	displacement vector		
T	tortuosity	3.5	90
t	time		
u	velocity of center of mass of fluid particle		
V	volume		
W	liquid content by weight	1.4.4	12

List of Symbols

w	subscript referring to wetting fluid		
wp	wetted perimeter	2.4	49
x,y,z	orthogonal space coordinates		
α	contact angle	1.5.1	16
γ	specific weight-- ρ g		
Δ	difference		
ε	empirical exponent	3.6.1	95
η	empirical exponent	3.6.1	95
θ	volumetric water content	1.4.4	12
λ	empirical exponent used as index of pore-size distribution	2.3.6	47
μ	dynamic fluid viscosity		
ρ	fluid density		
σ	interfacial force	1.5.1	16
τ	shear intensity	3.1.3	73
Φ	force potential	3.2	77
ϕ	porosity	1.3.1	3
ϕ_e	effective porosity	2.3.3	42
ψ	suction (negative pressure head)	1.5	13

Chapter 1

PROPERTIES OF POROUS MEDIA AND IMMISCIBLE FLUIDS

1.1. POROUS MEDIA

In its most general sense, the term "porous" could be applied to all matter, because all matter contains non-solid space. However, for the purpose of this text, additional restrictions are placed upon matter which is considered porous. These are:

(1) Non-solid space within the solid matrix is interconnected.

(2) The smallest dimension of the non-solid space must be large enough to contain fluid particles; that is, it must be large compared to the mean-free path of fluid molecules.

(3) Dimensions of the non-solid space must be small enough so that when interfaces between two fluids occur within the non-solid space, the orientation of interfaces is controlled largely by interfacial forces.

The first restriction eliminates consideration of a solid having only isolated pockets of non-solid space. It also eliminates consideration of a bundle of capillary tubes which are not cross-connected. A single capillary tube might be regarded as a porous medium, but not a bundle. The second restriction eliminates consideration of molecular transport through solids with non-solid spaces of such small dimensions that true convection cannot occur and fluid mechanics cannot be applied. The third restriction eliminates consideration of a network of pipes.

As an explanation of the latter point, consider the distribution of a mixture of water and air in a pipe compared to the distribution in a capillary tube, as illustrated in Figure 1.1.

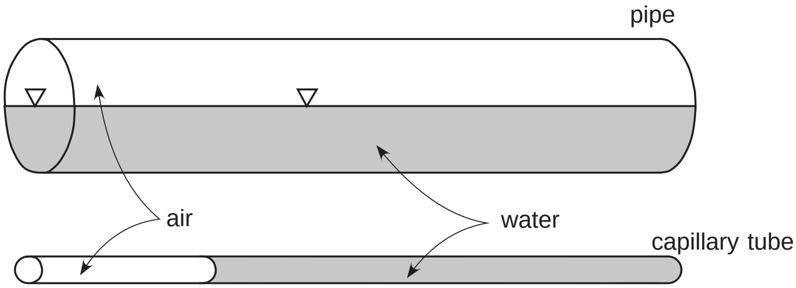

Figure 1.1. Orientation of interface in pipe as compared to the orientation in a capillary tube.

1.2. TYPES AND OCCURRENCE

Primary concern in this text is with flow in earth materials, e.g., soils and porous rocks. Most soils and porous rocks consist of porous materials that are granular in nature. That is, they are composed of relatively solid grains with non-solid space enclosed. The non-solid space is called pore space. Grains are sometimes cemented at points of contact with a variety of cementing agents and are said to be consolidated. A sandstone is an example of a consolidated porous medium. In other cases the grains are not cemented at points of contact and such materials are said to be unconsolidated. Most soils and aquifers are unconsolidated.

Another type of porous rock consists of non-solid space created by evolution of gases during crystallization and another by subsequent solution of soluble constituents in water. Examples of the latter are vugular limestones and dolomites. Pore space in such rocks consists of channels called vugs formed by solution. Not all porous limestones are vugular. Some consist of cemented fragments of more or less solid limestone. The latter are called intergranular limestones or intergranular dolomites.

Other types of porous material important in various fields include wood, living plant and animal tissue, textiles, building materials, filter materials and others.

1.3. CHARACTERIZATION OF PORE SPACE

Hydraulic behavior of fluids in porous media depends to a large degree upon the geometry of the non-solid space. In the case of granular material, geometry is influenced by the size and shape of individual grains and how they are cemented together.

Some granular media are said to possess structure. This implies that individual grains are cemented together to form porous aggregates that in turn enclose additional pore space. Pore space enclosed within individual grains is called primary pore space, and that enclosed between aggregates is called secondary pore space. A granular medium with structure contains a larger total volume of pore space in relation to its mass than one which does not possess structure.

Important characteristics of porous materials in relation to hydraulic behavior are described below.

1.3.1. Porosity

The average porosity of a sample of porous medium is defined as the ratio of interconnected pore volume V_p to the total volume V_t of the sample. The sample volume includes solid as well as non-solid space. In some literature, average porosity may be called simply porosity and designated by ϕ.

Sometimes, however, ϕ refers to a property which is assumed to apply to a point and varies in space. In this case, ϕ is defined as the ratio V_p/V_m in which V_p is the volume of pore space enclosed in V_m, a small volume of porous medium containing the point under consideration. To serve as a reference volume for this definition, V_m must be small compared to the entire system but large compared to individual solid grains. It must be sufficiently large that the ratio V_p/V_m does not change abruptly when a slightly larger reference element is considered, unless the larger element encompasses a boundary between different types of media.

In some analyses, ϕ is treated as if the ratio V_p/V_m is equal to the derivative of V_p with respect to V_m. Clearly an actual derivative is equal to either 0 or 1 depending upon whether the point under consideration is located within a solid grain or within the pore space, but this is not what is meant by ϕ. Consequently, the interpretation of ϕ as the derivative of V_p with respect to V_m requires a special definition of the derivative as the limit of V_p/V_m as V_m approaches a critical size somewhat larger than that at which the ratio V_p/V_m undergoes abrupt changes as V_m changes. Abrupt changes in the ratio occur, for example, if V_m is of a size of the same order of magnitude as that of individual grains.

Porosity at a point, therefore, is an abstraction that cannot be measured experimentally. Only the average porosity of a sample of rock or soil can be measured. This may be accomplished in one of several ways, some of which have been described by Collins (1961). Some of the methods determine the volume of liquid required to fill the pore space. Others determine the volume of the solid and this is subtracted from the total volume to obtain the pore volume. Still others utilize Boyle's law to compute the pore volume after allowing gas in the pores to expand. Factors affecting porosity include:

(1) **Structure -** Media with structure have larger porosities than media without structure.

(2) **Shape of grains -** A medium consisting of flat platelets can be packed so that the porosity is much smaller than for a collection of spheres. However, it would be possible to stack the platelets so that the porosity would be much greater than for spheres.

(3) **Grain-size distribution -** A medium consisting of spheres of varying sizes normally has a smaller porosity than one consisting of spheres of a single size. The smaller spheres may fit into spaces between larger spheres, thus reducing the porosity.

(4) **Mixing -** A medium consisting of two different sizes of spheres, for example, with the two sizes segregated into different regions has the same porosity as a single-size medium. If the two sizes are mixed, the porosity is reduced.

(5) **Packing -** The way individual grains are arranged can be influenced by mechanical conditions at the time of packing, for example, whether the particles settled out of water or were deposited by wind, or were deposited by some other geological process. Laboratory samples can be affected in this regard by the mechanical manipulation used in packing. The effect can be visualized by considering a stack of cards with the non-solid space enclosed when they are in a usual deck as compared to the space enclosed when they were arranged in a cubicle pattern.

(6) **Cementation -** The volume of cementing material, which may have precipitated from solution after the particles were deposited, reduces the porosity. Consequently, consolidated sandstones typically have smaller porosities than unconsolidated sand deposits.

Approximate porosities that could be expected in soils and porous rocks under various conditions include:

Consolidated sandstones	0.10 - 0.30
Uniform spheres packed to theoretical minimum porosity	0.26
Uniform spheres with normal packing	0.35
Unconsolidated sands with normal packing	0.39 - 0.41
Soils with structure	0.45 - 0.55

1.3.2. Specific Surface

The ratio of internal solid surface area to the total volume is called specific surface and is designated by s. In some texts the ratio of surface area to the mass of solid matrix is called specific surface. Specific surface may also be regarded as existing at a point, employing a concept analogous to that used for porosity.

The ratio of internal solid surface area to the total volume of a small element of the medium is defined as the specific surface of a point contained in the element. The volume element to be considered must be small relative to the volume of the system being analyzed and large enough that a slight increase in the volume would not produce a significant change in the ratio evaluated. In contrast to porosity, which is dimensionless, specific surface has the dimension of L^{-1}.

As in the case for porosity, specific surface is not something that can be determined experimentally at a point. A method of determining specific surface for a porous medium sample has been described by Bower and Goertzen (1959). With their method, a dry sample is allowed to adsorb ethylene glycol molecules from a saturated solution until a monolayer has formed on the solid surface. From the gain in weight of the sample produced by the monolayer, the surface area of the solid is computed. A method of determining specific surface by the adsorption of nitrogen gas has been described by Donaldson et al. (1975).

In the case of granular material consisting of grains of relatively uniform size, specific surface may be estimated from hydraulic measurements which are described in Chapter 3.

Specific surface is affected by the size and shape of individual grains, and to a lesser extent by structure. The smaller and flatter the particles, the greater the specific surface. Clay (which belongs to a class of minerals known as hydrous alumina silicates and consists of tiny plate-like crystals) has an enormous specific surface compared to other earth materials. Consequently, by far the most important factor (in determining the specific surface of earth materials) is the amount and types of clay in the medium.

Three important types of clay are recognized. These differ somewhat in chemical composition and there are many sub-types differing in respect to crystalline form. The three main types are: kaolinite, montmorillonite, and illite. Of the three, montmorillonite has the largest specific surface, illite the next largest and kaolinite the smallest. Expressed as square meters per gram, typical values are:

montmorillonite	800
illite	175
kaolinite	45

1.3.3. Pore Size

An average pore size $\bar{d}$ for a porous medium is defined as the ratio ϕ / s. The use of the term "average", in this case, does not mean that an entire sample of porous medium is considered. In fact $\bar{d}$ may be conceptualized, but not measured, as a property applicable at a point since both ϕ and s may be applicable at a point. When the reference volume used to define the average pore size is of the order of individual grain sizes it is designated here simply as d and may vary greatly from point to point.

Pore size has the dimension of length and is physically analogous to hydraulic radius as the latter term is used in hydraulics. To visualize this analogy, consider a section of a capillary tube, and note that the cross-sectional area of the tube divided by its internal perimeter is equal to the ratio of its volume to its internal solid surface, provided the cross-sectional area is uniform. If the cross-sectional area is not uniform, the ratio of volume to surface area gives the average hydraulic radius of the section.

Pore size is associated with grain size and grain-size distribution. The smaller the grains the smaller is the pore size. However, materials with structure may have some large pore sizes associated with the secondary pore space, even though the primary pore space is characterized by a small pore size.

1.3.4. Pore-size Distribution

Distribution of pore sizes is a concept applicable to d, but not to $\bar{d}$. The fraction of pore space represented by various ranges of d (within a volume element) has an effect of equal importance to the average magnitude of d. The distribution of d cannot be measured directly, but there is a way of defining and measuring an index of pore-size distribution. This is described in Chapter 2 where the concept of pore-size distribution and its effect on fluid flow is discussed in detail.

To some extent pore-size distribution is related to grain-size distribution. A wider distribution of grain sizes result in a wider distribution of pore sizes, other things being equal. Thorough mixing, however, may produce a relatively uniform pore size even with a wide distribution of grain sizes. A wide range of pore sizes is obtained in granular material only if it possesses structure.

1.3.5. Stability and Other Characteristics

The degree to which the characteristics described above remain constant in the presence of fluids (and under mechanical forces) is called stability. Both mechanical and chemical stability are important characteristics of porous media.

By far the most important stability factor for earth materials is the effect on pore-size distribution produced by the sensitivity of clay crystals to water. Clay crystals are usually stacked together like cards in a deck, the individual plates being bound together tightly in some cases and less tightly in others. The effect of water is to spread the plates, a process called clay swelling or dispersion. The opposite process, shrinking, results from flocculation of the clay crystals.

When clay is dispersed, larger pore sizes are eliminated and the range of the pore-size distribution is reduced. This has an enormous effect on the hydraulic properties of the medium which is explained in Chapter 3. Of the three types of clay, montmorillonite is the most sensitive to water solutions, kaolinite is least sensitive, and illite is intermediate. The order of sensitivity parallels that for the specific surface of the three clay types.

Water has less tendency to enter the space between clay plates when the solution is highly concentrated with electrolytes. Furthermore, certain ions inhibit swelling and dispersion much more than others. There is a tendency for ions of highest positive valence to be most effective. Thus, Al^{+++} is more effective than Ca^{++} and Ca^{++} is more effective than Na^{+} or H^{+}. Potassium, K^{+}, is an exception in that because of its particular ionic size, it inhibits swelling more than other monovalent ions.

In general, earth materials (rocks or soils) are more stable in the presence of hydrocarbon liquids than in the presence of water, especially if they contain significant quantities of clay or organic matter. The reason is that hydrocarbons have much less tendency to produce clay swelling. This is another reason why oils are often used in laboratories for flow experiments.

Mechanical manipulation of unconsolidated materials, especially when wet, also has an effect on the stability of unconsolidated media. The tendency is for mechanical manipulation to break down aggregates and reduce the size of larger pores. The shape of pore space and the degree to which it is interconnected are also important. Homogeneity and isotropy are important properties, but these are best characterized in respect to flow behavior and are defined in that context in Chapter 3.

1.4. FLUIDS IN POROUS MEDIA

Fluids belong to a class of matter the boundaries of which depend upon the geometry of the solid within which it is enclosed. This dependence results from the inability of fluids to support shearing stresses without continuous deformation.

The magnitude of shear stresses in fluids is a function of the rate of deformation of fluid elements. In some fluids, the magnitude of shear stress is proportional to the rate of deformation. Such fluids are called Newtonian viscous fluids and this text deals almost exclusively with these. Air, water and most petroleum fluids can usually be regarded as Newtonian without serious error. Although this text is restricted to a consideration of Newtonian viscous fluids, mixtures of fluids such as air and water, gas and oil or water and oil are considered.

1.4.1. Continuum

All fluids consist of particles such as molecules or ions, but this aspect of fluids is ignored in fluid mechanics by an artifice known as the continuum concept. Neither the properties nor motion of individual molecules are described by fluid mechanics.

The word "continuum" is related to the mathematical concept of continuous functions. A function f(x) is said to be continuous at a point $x = x_o$ if lim f(x) = $f(x_o)$ regardless of how x approaches x_o. The foregoing analytical definition of continuity is a formulation in mathematical language of the intuitive concept of continuity; that is, if the function f(x) is represented by a graph, and if it is continuous, the graph will have no breaks in the interval within which the function is continuous.

For fluid mechanics, the pertinent functions are the relationships between the space coordinates and the variables: pressure, velocity, and density. If the fluid is to be regarded as a continuum, it must be possible to define each variable (at every point within the region under consideration) as a property of a very small element containing the point. In addition, it is assumed that derivatives with respect to the space coordinates (at least the first and second derivatives) exist at every point. With some exceptions, fluid mechanics also assumes that pressure and velocity are differentiable to any order. In other words, it is usually assumed that the latter variables are analytic as well as continuous functions of the space coordinates.

One physical implication is that, theoretically, the fluid volume can be divided indefinitely without changing its basic character. This is obviously not realistic because all fluids consist of molecules between which there is empty space. The definitions of density, velocity, and pressure clearly are meaningless in respect to volume elements located in spaces between molecules.

The unrealistic assumption upon which fluid mechanics (a branch of continuum mechanics) is based, does not lead, necessarily, to significant errors in its application. For most cases in which fluid mechanics is applied, the assumption that the fluid is a continuum is entirely adequate because the dimensions of the fluid systems usually considered are large compared to the average distance between individual fluid molecules.

However, it is not always possible to define pressure, density and velocity as properties of a small volume containing a point in porous media, without the reference volume approaching the cube of the pore size. When the fluid under consideration is gas at atmospheric pressure, for example, the use of a continuum analysis leads to large errors for "fine-grained" media.

The continuum assumption is useful because it permits a description of the physical behavior with differential expressions. Without such an assumption, the analyses ordinarily performed by the methods of fluid mechanics becomes very cumbersome. However, the continuum assumption (in the sense it is used in fluid mechanics) cannot be applied to immiscible fluids separated by interfaces across which pressure discontinuities exist. In the latter case it is necessary to consider each individual fluid phase as a separate continuum.

1.4.2. Fluid Elements

In fluid mechanics the physical variables (including velocity) are specified at points within the fluid system. The variables are defined in respect to small volumes of fluid containing the points under consideration. A reference volume must be small compared to the pore size, but it must not be smaller than a limiting size which can be understood by referring to Figure 1.2.

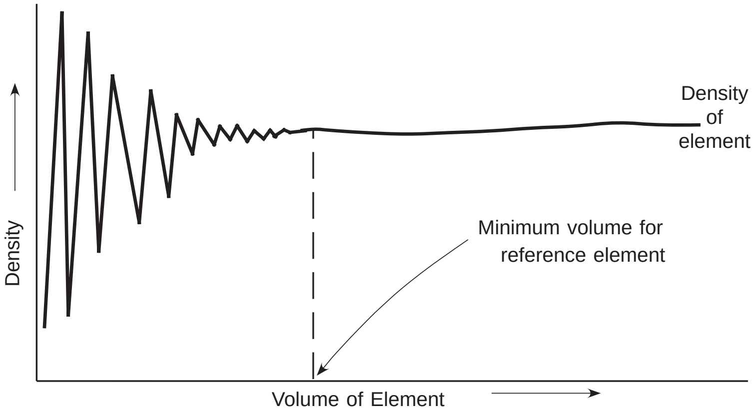

Figure 1.2. Density as a function of element volume.

Figure 1.2 shows a plot of hypothetical data that might be obtained by determining the density of volume elements of a fluid in the form of small cubes. If the dimensions of the elements were of the same order as the mean free path of the fluid molecules (or smaller), the density would vary erratically and be greatly different from one instant to another. The density variation would result from the random motion of the molecules and would depend on the number of molecules that happened to be in a particular volume element at a given instant. As the sample size is increased, the variation would decrease. At some critical dimension, large compared to the mean free path, the variation is negligible and a small increase in size of the element does not change the measured density significantly.

An element at least as large as the critical size, but small compared to the fluid system as a whole, is sometimes called a fluid particle. The critical size is much larger for gases than for liquids. No practical problems involving liquids are likely to be encountered for which the size limitation of a fluid particle invalidates a continuum assumption. For gases, on the other hand, many engineering problems can not be analyzed using the continuum assumption without introducing significant error. An example of the latter is the flow of gas in fine-grained porous media at atmospheric pressure.

1.4.3. Two-phase Fluid Systems

When two or more fluids exist within pore space, they are separated by boundaries (called interfaces) across which discontinuities in density and pressure exist. The existence of interfaces is characteristic of immiscible fluids. In the case of miscible fluids, there is no distinct boundary, at least on a microscopic scale. An example of the latter situation is a groundwater aquifer into which salt water has intruded. On a macroscopic scale it is possible to find distinct regions of salt and fresh water, but on a microscopic scale there is no interface as this term is used here.

An interface is made possible by the existence of forces (called interfacial forces) that act only at boundaries between separate phases and are tangential to the boundaries. When boundaries are curved, as they are in porous media, the tangential interfacial force produces pressure discontinuities at the interfaces.

A phase may consist of a number of chemical constituents, but each phase is assumed to be homogeneous within itself and to constitute a physical continuum to which the mathematical methods of fluid mechanics can be applied. Fluid mechanics (in the sense this term is used here) cannot be applied to a mixture of phases.

1.4.4. Liquid Content

In the case of a two-phase system such as a liquid and gas, the volume fraction of the total pore volume (of a medium element) occupied by liquid is called saturation and is designated by the symbol S. Saturation can be conceptualized (but not measured) as a point property varying in space in a manner entirely analogous to porosity. Saturation should not be confused with the term "saturated" which means that only a liquid phase exists, or in other words, S is equal to unity.

Another way of expressing liquid content is as a fraction of the volume of porous medium in which it is contained. The latter variable is called volumetric water content by soil physicists and is often designated as θ. A third method is as a fraction W of the dry weight of porous solid in which the liquid is contained. Designation of liquid content by W is found most often in the literature of agronomists and soils engineers.

The relationship among the three expressions for liquid content is

$$\theta = S\phi = \gamma_s \left(1-\phi\right)W, \tag{1.1}$$

γ_s being the specific gravity of the solid. The expression for liquid content used depends upon the application under consideration. In this text, each of them is used where appropriate.

Petroleum scientists frequently must consider systems in which three fluid phases, such as, brine, oil and gas, are contained in the pore space of porous rocks. In their literature, the symbols S_w, S_o and S_g may designate saturations of brine, oil and gas, respectively, each representing a fraction of the total pore volume of the reference element.

Wherever two or more fluid phases occupy a porous medium, one of the fluids is adsorbed on the solid surfaces more strongly than the other fluid or fluids. The fluid which is most strongly adsorbed (and which displaces the others from the adsorbed film) is called the wetting fluid or wetting phase. The displaced fluid is the nonwetting phase. When referring to the wetting phase, the fluid properties such as pressure or velocity are designated as p_w and u_w, etc. Similarly, the subscript nw designates the properties of the nonwetting phase.

In most cases, liquids are adsorbed more strongly than gases and as a consequence, in a two-phase system involving a liquid and gas, the liquid usually is the wetting phase. An exception is the case of a mercury-gas system for which mercury is the nonwetting phase. If the term "saturation" is used without adjectives to describe fluid content, it usually refers to a volume fraction of the wetting phase in a two-phase system. The content of other phases is usually referred to as "gas saturation" or "nonwetting phase saturation," etc.

1.5. CAPILLARITY

At boundaries between phases, forces of cohesion between fluid molecules are not, within themselves, balanced. There is a force component from unbalanced cohesion which acts tangentially to the boundary and which is called interfacial force. Discussions of the origin of interfacial force have been presented by Adam (1968) and the Encyclopedia Britannica (1964), among others.

Interfacial force acts in a direction tending to contract the interfacial area, in a manner somewhat analogous to tension in a stretched membrane. For this reason, it is often called surface tension, the dimensions being force per unit length or energy per unit area. The analogy between interfacial force and the tension in a stretched membrane is not complete, however, because in the case of the former, the force is unrelated to deformation. Interfacial force is a function only of the physical and chemical properties and state of the two fluid phases in contact.

The failure, in some respects, of the analogy between interfacial force and tension in a membrane has led some authors to reject the concept of interfacial force as a physical reality. They prefer to speak only of interfacial energy. This view does not seem to be tenable, however, considering the fact that energy can only be defined in terms of force fields, and if there is no unique force at points in the interface, there could be no energy uniquely associated with the interface.

The resultant of interfacial force acting on a curved interface is balanced, at equilibrium, by a difference in pressure at points of contact between fluid phases. Without interfacial force, separate phases would not exist; that is, distinct interfaces would not exist. The difference in pressure, called capillary pressure, is designated by p_c and is defined by

$$\mathbf{p_c \equiv p_{nw} - p_w} \,, \qquad 1.2$$

In soils literature, p_c is sometimes called suction or matric suction and designated by ψ. However, this symbol is more often used to indicate the negative pressure head of water in a water-air system, assuming soil air is everywhere at zero gauge pressure. In the latter case, ψ is $p_c / \rho_w g$ and may also be called soil water tension. Suction, capillary pressure and tension are closely related but not identical variables.

One effect of interfacial force is a tendency to compress the nonwetting phase relative to the wetting phase. For example, in a water-gas system, water tends to be preferentially adsorbed by the solid surfaces and the gas is compressed if it is entirely surrounded by water. The gas pressure, therefore, is higher than the water pressure at points of contact. In a water-oil-gas system, the gas usually is at the highest pressure, the oil at the next highest and water at the lowest pressure when equilibrium exists.

In a system in contact with the atmosphere, such as a soil, either the wetting phase (water) or the nonwetting phase (air) is normally at atmospheric pressure. If the soil is sufficiently desaturated for an interconnected air phase to exist, the air is at atmospheric pressure and the liquid is at less than atmospheric pressure. This is the reason soil scientists refer to water as being under "suction" or "tension." On the other hand, if the soil is in contact with water at atmospheric pressure, that is, if it is flooded, some air may be entrapped within the pore space below the level where water is at atmospheric pressure, and this air is at a pressure greater than atmospheric.

1.5.1. Factors Affecting Capillary Pressure

The difference in pressure between phases occupying the pore space of a porous medium is related to gravity, saturation, pore-size, pore-shape, interfacial forces, the angle at which fluid-fluid interfaces contact solid surfaces, the density difference between phases and the radii of curvature of interfaces. These factors are not all independent variables in respect to their effect on capillary pressure. There is, however, an interrelation among them.

The way that gravity and saturation are related to capillary pressure is discussed in Chapter 2. The relationship among capillary pressure and the other variables mentioned above can be visualized by considering the following analysis in reference to Figures 1.3, 1.4 and 1.5.

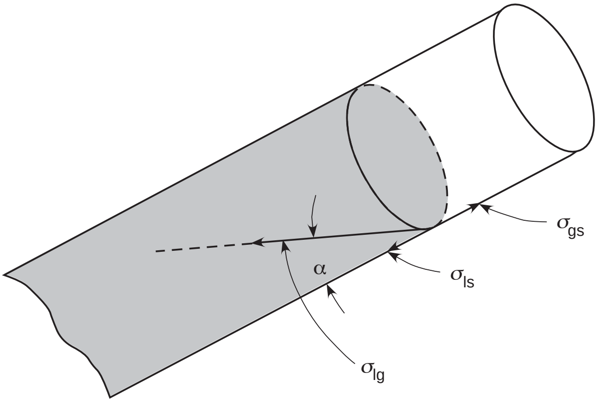

Figure 1.3. Liquid-air interface across a capillary tube.

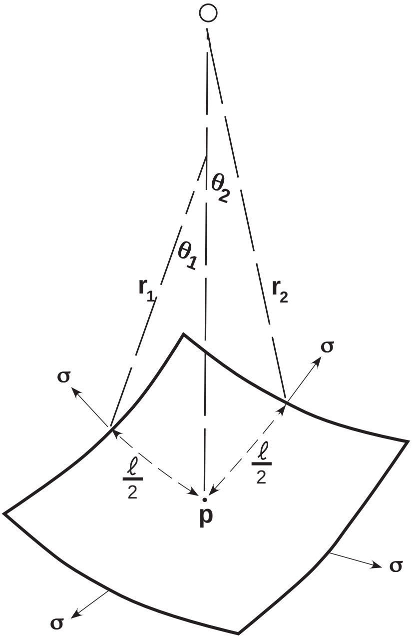

Figure 1.4. Relationship between capillary pressure and curvature at a point.

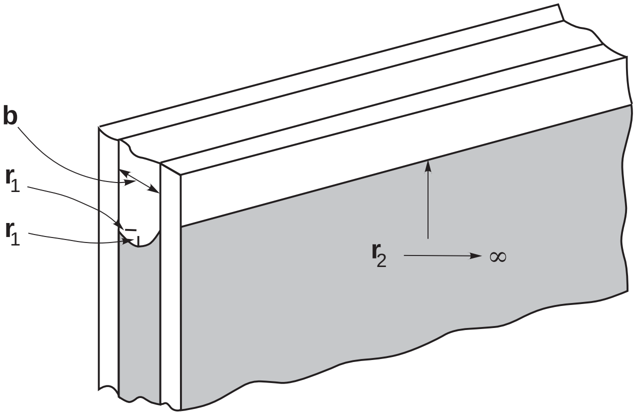

Figure 1.5. Interface across space between parallel flat plates.

First, a balance of forces is considered on a free body consisting only of the line of contact of the fluid-fluid interface with the solid (see Figure 1.3). There are three interfacial forces $\boldsymbol{\sigma}$ acting on the line of contact. These are:

(1) liquid-solid $\boldsymbol{\sigma}_{\ell \mathbf{s}}$,

(2) gas-solid $\boldsymbol{\sigma}_{\mathbf{gs}}$,

(3) liquid-gas $\boldsymbol{\sigma}_{\ell \mathbf{g}}$.

Surface forces act at all points along the line of contact in a direction indicated in Figure 1.3. However, gravity and pressure do not act on this free body because it has neither mass nor area. Also, the component of $\sigma_{\ell g}$ normal to the solid surface is balanced by the normal force of the solid on the line of contact. The balance of forces, therefore, is given by

$$\mathbf{2\pi r_t}\left[\boldsymbol{\sigma}_{\mathbf{gs}} - \boldsymbol{\sigma}_{\ell \mathbf{s}} - \boldsymbol{\sigma}_{\ell \mathbf{g}}\,(\mathbf{cos}\,\boldsymbol{\alpha})\right] = \mathbf{0} \qquad 1.3$$

where r_t is the radius of the tube and α is the contact angle of the fluid-fluid interface with the solid.

Equation 1.3 is useful to show, in a qualitative way only, how the contact angle is related to the three interfacial forces. The contact angle cannot actually be computed from Equation 1.3 because $\boldsymbol{\sigma}_{\mathbf{gs}}$ and $\boldsymbol{\sigma}_{\ell \mathbf{s}}$ cannot be measured. However, the contact angle must adjust so that the component of $\boldsymbol{\sigma}_{\ell \mathbf{g}}$ (in the direction of the axis of the tube) is equal to the difference $\boldsymbol{\sigma}_{\mathbf{gs}} - \boldsymbol{\sigma}_{\ell \mathbf{s}}$.

Next, a balance of forces is considered on a free body consisting of the interface as a whole. In this case, the pressure difference across the interface must be considered, but gravity does not, because the interface has area but no mass. The unmeasurable interfacial forces, can be eliminated from consideration by considering (as a free body) that part of the interface which is not in direct contact with the solid. The balance of forces is further simplified by considering only the components of the pressure forces normal to a plane passing through the line of contact of the interface with the solid.

With these simplifications, the balance of forces is given by

$$\bar{\mathbf{p}}_{\mathbf{c}}\left(\boldsymbol{\pi}\mathbf{r}_{\mathbf{t}}^{\mathbf{2}}\right) = \boldsymbol{\sigma}_{\ell \mathbf{g}}\,\mathbf{cos}\,\boldsymbol{\alpha}\left(\mathbf{2\pi r_t}\right)$$

or

$$\bar{p}_c = \frac{2\sigma}{r_t}\cos\alpha \qquad 1.4$$

where $\bar{p}_c$ is the average capillary pressure over the interface.

Equation 1.4 shows that capillary pressure varies inversely with the radius of the tube across which an interface is positioned and directly with the cosine of the angle of contact. However, capillary pressure varies from point to point over the interface because of gravity. This variation is associated with a corresponding variation in curvature of the interface with elevation over the interface.

The dependence of capillary pressure on curvature is analyzed with reference to Figure 1.4 which represents a small segment of a curved interface containing the point p. The point is at the center of an approximately square segment. Edges of the segment are each of length ℓ. The angles θ_1 and θ_2 are those subtended by an arc length of $\ell/2$ in orthogonal planes normal to the segment at p, with radii of curvature r_1 and r_2 respectively.

For small θ,

$$\sin\theta_1 \approx \frac{\ell}{2r_1} \cdot$$

Balancing the force components, pressure and interfacial, normal to the segment at p results in

$$p_c\ell^2 \approx 2\ell\sigma\left(\sin\theta_1 + \sin\theta_2\right)$$

or

$$p_c \approx \sigma\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \cdot$$

In the limit, as ℓ approaches 0, the relationship becomes exact,

$$p_c = \sigma\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \cdot \qquad 1.5$$

Equation 1.5 is called the Laplace equation of capillarity. It evaluates the pressure difference across the interface at a point as a function of surface tension and curvature at the point only. The contact angle and pore size are not involved explicitly.

It is informative to consider the relationship between Equation 1.4 and 1.5 by applying Equation 1.5 to an interface across a tube as illustrated in Figure 1.3. For a very small tube such that fluid weight has negligible effect, capillary pressure is constant over the interface consisting of a segment of a sphere. For example, with a zero contact angle, the interface is a hemisphere with radii of curvature equal to the radius of the tube. Since for this case radii of curvature are equal, Equation 1.5 reduces to the same form as Equation 1.4. If the angle of contact is not zero, equating Equation 1.4 to Equation 1.5 shows that the radius of curvature of the interface is the radius of the tube divided by $\cos\alpha$. However, this result assumes the absence of a gravitational effect.

The quantity $(1/r_1 + 1/r_2)$ is sometimes referred to as the mean curvature of an interface at a point. Mean curvature is affected by pore shape as well as pore size. The effect of shape can be visualized by considering an interface across a pore space with a shape that is different from that of the tube illustrated in Figure 1.3. In the case under consideration, shown in Figure 1.5, the interface is positioned across the space between two flat parallel plates.

The tendency is for an interface to reach (at equilibrium) an orientation, position and shape such that the total potential energy of the system (relative to the force fields of gravity, interfacial energy and pressure) is a minimum. In the case under consideration, this tendency results in an interface that is horizontal in vertical planes parallel to the plates. This is produced by the dominant effect of gravity in controlling the curvature in parallel planes, the dimension of the fluid body being large in this direction. However, in vertical planes normal to the plates, if the spacing "b" is small, $r_1 = b/(2\cos\alpha)$. For this case Equation 1.5 reduces to

$$\mathbf{p_c = \frac{2\sigma \cos\alpha}{b}} \; . \qquad 1.6$$

If the dimension of the pore space parallel to the plates is relatively large, say more than a few millimeters, the magnitude of capillary pressure is unaffected by further increases in this dimension. Capillary pressure in such cases is controlled only by the small dimension normal to the plates.

In the case of interfaces in pore spaces enclosed in granular porous solids, the orientation of interfaces is such that the area of interface tends to be a minimum consistent with the fraction of wetting or nonwetting fluid contained in the space.

1.5.2. Factors Affecting Surface Tension

Interfacial force is determined by chemical properties of the two phases in contact, especially chemical properties of the boundary layer of the two phases. It is also affected by the temperature and pressure of the two phases.

Since surface tension is sensitive primarily to the chemical properties of boundary layers, any chemical constituent tending to accumulate preferentially in the boundary layer, may have a large effect even though the average concentration in the phase as a whole may be very small. Such agents are said to be surface active and are called surfactants. Surface active agents which reduce surface tension for water, for example, occur frequently as contaminants in soils, rocks and in laboratories. Most often they consist of organic molecules.

The reasons why many organic molecules are surface active can be understood by considering the following principles:

(1) The tendency of the system as a whole is to reach a condition of minimum energy. Other things being equal this means a minimum surface energy.

(2) In a solution, the tendency is for molecules or ions to mix homogeneously, but the effect of interfacial energy is to oppose this tendency.

(3) With a mixture of constituents in a solution, surface tension is determined primarily by the molecules actually in the surface.

(4) Molecules or ions in the surface layer are under the influence of interfacial force. Therefore, they possess more energy than interior molecules.

(5) It requires energy to move molecules from the interior to the surface.

(6) With a mixture of constituents, the minimum surface energy is obtained when constituents which result in a lower surface tension have a higher concentration in the surface.

Based on the above considerations, J. Willard Gibbs (1876) derived an equation relating change in surface tension to concentration of particular chemical agents. He showed that

$$\frac{d\sigma}{dc} = -RT\frac{\Delta c}{c} \qquad 1.7$$

in which

Δc ≡ excess concentration of agent in surface layer,

c ≡ concentration of agent in bulk of liquid phase,

R ≡ gas constant,

T ≡ absolute temperature.

Equation 1.7 reflects the fact that an agent tending to concentrate in the surface, so that Δc is positive, causes the surface tension to decrease with respect to c. Therefore, an agent tending to reduce surface tension accumulates preferentially in the surface, because by accumulating largely in the surface, the surface tension is reduced in conformance with the principle of minimum energy at equilibrium.

An agent tending to increase surface tension results in a smaller concentration in the surface layer than in the remainder of the fluid in conformance with the same principle, and Δc is negative. Since only constituents in the surface layer have a significant effect on surface tension, the concentration c must be larger to produce a substantial effect in the latter case. Consequently, agents that tend to increase surface tension cannot be called surfactants. Most inorganic salts are in a class of chemicals that produce a small increase in surface tension of water solutions in contact with air.

Surfactants usually consist of relatively large molecules, often organic. Large polar molecules that tend to orient themselves to exclude other molecules in the surface are particularly effective in this respect. Many such materials consist of organic salts of inorganic ions. An example is sodium sulfonate. The Na+ ion tends to dissolve in a water solution leaving a negatively charged sulfonate group (anionic) oriented in the surface with the negative end pointed toward the interior of the solution. Other surfactants are cationic, that is, the surface active part possesses a positive charge, and some are non-ionic.

Almost any organic contaminant reduces the surface tension of pure water. In fact, the surface tension of pure water is very unstable. This is because the surface tension of water is greater than the surface tension of most common liquids. Consequently, the value of surface tension for water in soils and porous rocks is less than the surface tension specified in handbooks for water at a particular temperature. Water in naturally occurring earth materials often has a value of $\sigma \cos\alpha$ of about 60 dyne/cm, whereas for pure water in contact with air at 20˚C the value is about 72 dynes/cm. In contrast, most oils have a smaller surface tension so they are not so easily affected by common contaminants. For this reason, oils are often used in place of water in laboratory studies.

Temperature also affects surface tension. The variation of surface tension of water in contact with air at room temperatures is small, being about 75 dynes/cm at 5˚C and 72 dynes/cm at 25˚C. At higher temperatures, the change of surface tension in respect to temperature increases, and at 100˚C, the surface tension of water-air is about 50 dynes/cm. Some representative examples of other approximate surface tensions in dynes/cm at 20˚C include:

mercury-air	470,
mercury-water	375,
water-octane	51,
ethyl alcohol-air	21.5,
water-octyl alcohol	8.5.

1.5.3. Wettability

Interfacial force also exists at solid-fluid interfaces, although it is extremely hard to measure. The interfacial force at solid surfaces is important, because it controls which fluid is the wetting or the nonwetting phase. It also determines the contact angle of the interface with the solid as Equation 1.3 implies. An associated property of solid surfaces is the energy required to remove adsorbed constituents of various fluids, that is, the strength of adhesion.

Such properties are one aspect of what is called wettability. Another aspect of wettability is the speed with which fluids spread over solid surfaces. The speed of spreading is affected by the surface tension and also by viscosity. Spreading speed is increased by lower surface tensions and lower viscosities.

When water and oil are introduced at the same time into a porous medium, the surface is first wet by the oil and later oil is replaced in the adsorbed layer by water. The initial wetting by oil is due to its smaller surface tension in contact with air resulting in oil spreading faster than water. The reason for oil being eventually replaced by water is that water is more strongly adsorbed.

The chemical condition of solid surfaces, particularly in respect to adsorbed organic contaminants, is important in determining the strength of adhesion of particular fluids and, to some extent, the speed of spreading. Petroleum scientists frequently classify porous rocks as being either water-wet or oil-wet, depending on the liquid preferentially adsorbed on the solid. Given sufficient time, water replaces most oil constituents on the solid surfaces of nearly all porous rocks. The condition of being oil-wet is usually a temporary situation existing in laboratory samples after they have been dried and have a residue of adsorbed organic contaminants remaining. Most contaminants can be removed by heating the samples to the ignition temperature of the contaminant.

Some adsorbed organic molecules, however, are very difficult to remove except by burning. Bradford sandstone in Pennsylvania is a well-known example of rock considered to be oil-wet in its natural state. In some cases, porous materials may be deliberately made water-repellant by the adsorption of agents such as silicon on their surfaces.

Agricultural, lawn, and forest soils sometimes become temporarily water-repellant after desiccation because of the presence of organic substances, especially bitumens produced by the metabolic processes of certain microorganisms. Prolonged exposure to water often restores the normal hydrophilic character of soil. Exposure of soil surfaces to heat, for example from a torch, removes the bitumens faster.

PROBLEMS AND STUDY QUESTIONS

1. By knowing the specific gravity of sand particles, usually about 2.65, and the total volume in which a given weight of this sand is contained, describe how the average porosity of the unconsolidated sample could be estimated.

2. Could a similar method of estimating average porosity (as defined in this text) be used to estimate the porosity of a volcanic rock containing many void spaces formed from gas bubbles at the time of lava cooling? Explain.

3. Explain why a geological formation consisting of cavernous limestone rock might not behave as a porous medium as defined in this text.

4. Agricultural soils usually have much greater porosities than deposits of sand on beaches. Explain.

5. Describe the effect on the average pore size of adding a relatively small amount of montmorillonite to a sample of beach sand. Explain.

6. Explain why it is appropriate to apply the term "average pore-size" for the ratio ϕ / s.

7. Would you expect the addition of montmorillonite to beach sand to have any significant effect upon the pore-size distribution? Explain.

8. Would you expect average porosity of a consolidated sandstone rock (containing a small amount of clay) to be affected to a greater or lesser extent by clay dispersion than a pore-size distribution index? Explain.

9. Explain why a "fluid particle" in a gas system is necessarily larger than a fluid particle in a water system.

10. Derive, by means of the "balance of work" principle, an equation for the excess pressure inside a spherical air bubble surrounded by liquid.

11. Determine the pressure intensity within a soap bubble 0.01 cm in diameter. Assume the surface tension of the soap solution, at air interfaces, is 40 dynes/cm.

12. Explain why the surface tension for the soap solution is 40 dynes/cm and not about 70 dynes/cm as would be the case for pure water.

13. Contrast the trends of pressure variation during the process of (a) blowing up a toy balloon, and (b) blowing a soap bubble. Explain.

14. In what physical way does an oil-gas two-phase system differ from a brine-fresh water system encountered in a salt water intrusion into a coastal aquifer? Explain.

15. Explain how the relative strength of adhesion to a solid surface of two fluids (mixed in a porous medium) determines which fluid phase is at the higher pressure.

16. Why is it necessary to consider a system of two immiscible fluids as two separate continua in the application of fluid mechanics?

17. A porous medium sample consisting of a liter of beach sand may have either a slightly larger or a slightly smaller porosity than a liter of uniform marbles. Describe what factors might make the porosity of the sand greater and what factors might make the porosity less than that of the marbles.

18. Agronomists often refer to water in a soil profile within the root zone of plants as being under "tension." Is this use of the term "tension" analogous to the use of this term in connection with the state of stress in a steel bar? Explain.

19. Would you consider the term "suction" more (or less) appropriate than "tension" to designate capillary pressure? Explain.

20. Would you expect a water-air interface or a water-oil interface to have a smaller angle of contact? Explain.

21. Explain the contrast in orientation of interfaces illustrated in Figure 1.1.

22. Considering the principle of minimum energy for equilibrium, would you expect (with a very small concentration of amyl alcohol) to find all of the alcohol molecules in the surface layer? Explain the principle involved in terms of thermodynamic concepts.

23. Give a possible reason why surface tension for mercury-air is greater than for mercury-water and why both are greater than surface tension for water-air.

24. It sometimes happens that in late summer, lawns develop dry spots that seem hard to wet. By poking a few holes through the turf in the dry spots, the soil can be made to absorb water easily. Explain.

25. Consider a cylindrical sample of porous sandstone (granular) having a diameter of 2.54 cm and a length of 6 cm. The sand grains have a specific gravity of 2.65. Before drying, the sample weight is 60 gm. After drying, the weight is 53 gm. Assume the liquid contained in the sample before drying was entirely water. Estimate ϕ, S, θ and W before drying.

26. In reference to the sample described in problem 25, describe a method of checking the estimated porosity using an independent procedure.

27. Given that the solid grains contained in the sample of porous sandstone of problems 25 and 26 have a specific surface measured by the ethylene glycol procedure to be 10 m^2/g, estimate the average specific surface of the bulk sample in cm^2/cm^3.

Chapter 2

IMMISCIBLE FLUIDS IN STATIC SYSTEMS

2.1. MECHANICAL EQUILIBRIUM

A fluid system is said to be in equilibrium when there is no net transfer of matter within the system and no flow of heat. The conditions for equilibrium may be studied from the point of view of thermodynamics. A thermodynamic description of a fluid system begins with a definition of a system that in this case is a definite quantity of fluid in interchange with the surroundings only by flow of heat or by doing work. In the case under consideration, the porous solid is the surroundings.

According to Zemansky (1943) there are three conditions that must be satisfied for any system to be in equilibrium:

(1) There must be no unbalanced driving force on any element of the system and none between the system and its surroundings. This is a condition for mechanical equilibrium.

(2) The fluid system must not undergo a spontaneous change of internal structure such as a transfer of matter from one phase to another. A system that meets this requirement is in chemical equilibrium.

(3) All parts of the system must be at the same temperature, and this temperature must be the same as that of its surroundings. Such a system is in thermal equilibrium.

When conditions for all three types of equilibrium are satisfied, the system is said to be in a state of thermodynamic equilibrium. States of thermodynamic equilibrium can be described in terms of macroscopic coordinates not involving time, that is, thermodynamic coordinates which refer to the system as a whole and not to its parts.

If an attempt is made to describe a fluid system in terms of thermodynamic coordinates, the system must be relatively small, otherwise the description is inadequate. Furthermore, when equilibrium does not exist, coordinates referring to a system as a whole may not be possible unless the system selected is very small. This is because gradients of pressure and temperature exist.

The approach used in fluid mechanics is to choose a system consisting of either a very small control volume that remains fixed in space or a small element of fluid that may move in space. The latter approach is employed here.

In order to limit the scope of phenomena considered, to an entity of reasonable size, only problems relating to mechanical equilibrium are considered in detail. The following analyses assume chemical and thermal equilibrium exist. A fluid is said to be static when all of its elements are in mechanical equilibrium. In such a state, none of its elements move with respect to a coordinate system fixed to the solid boundaries, because the driving forces on each element are balanced.

A fluid particle as defined in Section 1.4.2 is selected as a reference element (or free body) for the analysis of forces acting on a static fluid. A fluid particle is considered which is entirely within a single fluid phase. Consequently, there is no ambiguity in respect to density and other properties assigned to the particle. It is regarded as being a part of a single continuum constituting one fluid phase which may be mixed with another phase (or other phases) occupying the pore space.

In some cases, the nonwetting fluid may be divided into separate parts completely surrounded by the wetting phase. When a phase is discontinuous in this sense (not interconnected) it may be regarded as a part of the porous matrix, analogous to the solid grains, in an analysis of the wetting phase. When two or more interconnected phases are present, each is analyzed separately as a single continuum.

2.1.1. Forces on Static Fluid Particles

Forces acting on a reference particle are classified as driving or resisting forces according to whether they tend to produce motion or are a consequence of motion. In the case of static fluids, only driving forces act on fluid particles. Furthermore, their resultant must be zero. Driving forces, in turn, can be classified according to whether they are proportional to the mass of the particle or to the surface area of the particle. The former are called body forces and the latter surface forces.

The most conspicuous of the body forces acting on fluid elements is gravity. For the present analysis, it is assumed this is the only body force. Force per unit volume due to gravity is $\rho\mathbf{g}$ where ρ is fluid density and **g** is a vector representing force per unit mass due to gravity. The resultant of **g** is directed vertically downward through the center of mass of fluid particles. Although **g** varies inversely as the square of the distance between the fluid particle and the center of the earth's mass, it is sufficient to regard **g** as a constant for applications considered here. A magnitude of **g** of 980 dynes/gram is close enough for most purposes.

The force acting on the surface of fluid particles, in general, can have tangential as well as normal components. Tangential components, however, exist only if there is relative motion of the center of one fluid element in respect to another. This does not happen in a static fluid. The normal component of surface forces may be a result of two factors. The first factor called pressure or hydrostatic pressure is the normal component that can be related in an equation of state to the density and temperature of the fluid. This portion of the normal surface force is conservative, that is, it has a fixed value for a given fluid at a particular temperature and density.

In the case of static fluids, pressure is the only surface force, and it acts equally in all directions. From a microscopic point of view pressure results from the rate of momentum transfer (by fluid molecules) normal to any differential area at a point in the fluid. In static fluids, pressure is invariant in respect to orientation of the differential area. Pressure, therefore, is a scalar quantity acting at a "point." It must be defined, however, with respect to a volume element (the size of a fluid particle) containing the point. The definition is given by

$$\mathbf{p} \equiv \lim \frac{1}{\mathbf{A}} \int_{A} \boldsymbol{\sigma}_c \bullet \mathbf{dA} \qquad 2.1$$

in which A is the magnitude of the surface area of a fluid particle, **dA** is a differential surface vector, and $\boldsymbol{\sigma}_c$ is the conservative surface stress, positive outward. Note that $\boldsymbol{\sigma}_c$ is a vector quantity whereas p is a scalar.

Another part of the normal surface force results from viscous resistance to expansion or compression of fluid particles. It does not exist in static fluids or in flowing fluids undergoing no expansion or contraction, that is, zero divergence. The portion of normal surface stress that is not conservative, and is associated only with divergence, is not related to density at a particular temperature. In the case of flowing fluids, even those undergoing negligible divergence, surface stress is not, in general, invariant in respect to direction. In such cases Equation 2.1 is still a valid definition of p and is given by

$$\mathbf{p} = \frac{1}{3}\left(\sigma_i + \sigma_j + \sigma_k\right) \qquad 2.2$$

in which σ_i, σ_j, and σ_k are normal surface-stress components in three orthogonal directions. In this case, also, p is a scalar quantity, although it has the dimensions of force per unit area.

For the analysis of static fluids the only forces acting on fluid particles are gravity and the resultant of the pressure force. Because pressure is a scalar, the resultant force on a fluid particle due to pressure is due to the variation of pressure in space. Specifically, the component of force per unit volume due to pressure (in a particular direction i) is given by

$$-\frac{\partial p}{\partial x_i},$$

where x_i is a coordinate length in the direction i, and the minus sign indicates the force is positive in the direction i if p decreases in that direction.

A balance of force components in the direction i is given by

$$-\frac{\partial p}{\partial x_i} + \rho g_i = 0 \qquad 2.3$$

in which g_i is the component of gravity in the direction i. Note that the force components in Equation 2.3 are expressed as force/volume.

2.1.2. Forces on Fluid in a Control Volume

The preceding analysis is in respect to a reference element consisting of a fluid particle. Some authors prefer to consider a force balance on a fluid volume contained in a reference element such as is used in Section 1.3.1 for the definition of porosity. To show that the latter approach also leads to Equation 2.3, a simplified model of a reference element, as illustrated in Figure 2.1, is considered.

The reference volume V_m consists of both solid and fluid space, but only forces acting on the fluid contained in the reference volume are considered. In this case, a single fluid is assumed to occupy all of the pore space.

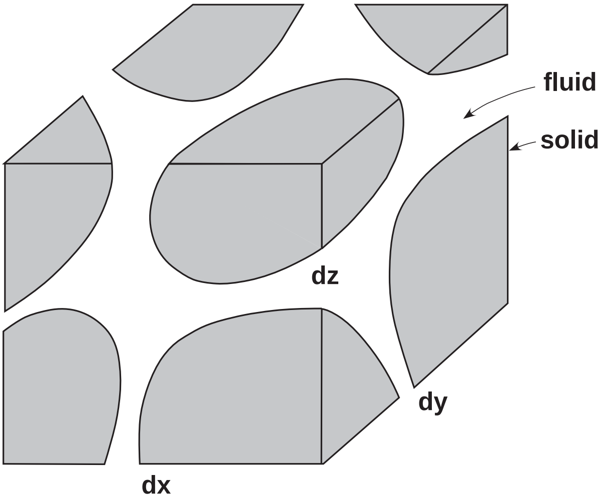

Figure 2.I. Cross-section of model for reference volume

Force due to gravity is $\rho g \phi V_m$, or if only the component in a particular direction i is considered, $\rho g_i \phi V_m$. The resultant force from unbalanced fluid pressure on the faces of the reference volume can be visualized by considering the forces existing on two opposite faces, say top and bottom. For this purpose, a situation is considered in which porosity may be varying in space so that the fluid area exposed at the upper face of the reference volume is ϕ_u dx dy, and at the lower face is ϕ_1 dx dy. The total force due to fluid pressure on the upper face is $p_u \phi_u dxdy$ and that on the lower face is $p_1 \phi_1 dxdy$. The net component of force in the positive z direction is

$$\left(p_1 \phi_1 - p_u \phi_u\right) dx\, dy .$$

Insofar as pressure and porosity can be regarded as point concepts, the component in the z direction due to fluid pressure on the reference element is

$$-\frac{\partial(p\phi)}{\partial z} dz\, dx\, dy \ .$$

which expressed as force/volume is

$$-\frac{\partial(p\phi)}{\partial z} .$$

In three dimensions, the fluid pressure force in any direction i is

$$-\frac{\partial(p\phi)}{\partial x_i} .$$

However, since the reference element is the fluid contained in the volume V_m, the solid surface as well as the fluid surface exerts a force on the fluid within V_m. If porosity of the medium is uniform so that the fraction of solid on each face is the same, the resultant of the solid force on the fluid element is zero.

This situation can be visualized by expanding the expression for the force/volume exerted by the fluid, that is,

$$-\frac{\partial p}{\partial x_i} + \rho g_i = 0$$

The first term on the left represents a force exerted by fluid on the fluid in the reference element. The second term on the left represents a force exerted by fluid on solid surfaces of the reference element. However, the force of fluid on solid surfaces is counterbalanced by an opposite force of solid surfaces on the reference fluid element. The net surface force/volume is given by

$$-\phi\frac{\partial p}{\partial x_i} - p\frac{\partial \phi}{\partial x_i} + p\frac{\partial \phi}{\partial x_i} = -\phi\frac{\partial p}{\partial x_i} .$$

Summing the pressure force and the gravitational force results again in Equation 2.3.

2.1.3. Adsorptive Forces

The foregoing analyses of forces on static fluid particles assume the only body force of significance acting on the particles is gravity. Edlefsen and Anderson (1943) have shown this is not necessarily a valid assumption, especially in reference to a wetting phase at low saturations.

Forces at fluid-solid interfaces producing adsorption and wetting may extend outward from the solid surface for a few hundred molecular layers. Although such forces are known to be very strong, they are also very short ranged. It has been estimated that they vary inversely with approximately the 5th power of distance from the solid surface. If a particle is more than say 100 molecular diameters from a solid surface, the adsorptive force can probably be neglected. But, the potential energy due to adsorptive force cannot necessarily be regarded as small.

The magnitude of adsorptive force, that may be electrostatic in origin or due to a variety of other possible factors, depends upon the species of molecules in a fluid particle. It is not entirely clear such forces can always be treated as a body force, but following the precedent of Edlefsen and Anderson (1943), adsorptive force is treated as a body force on the fluid particle illustrated in Figure 2.2.

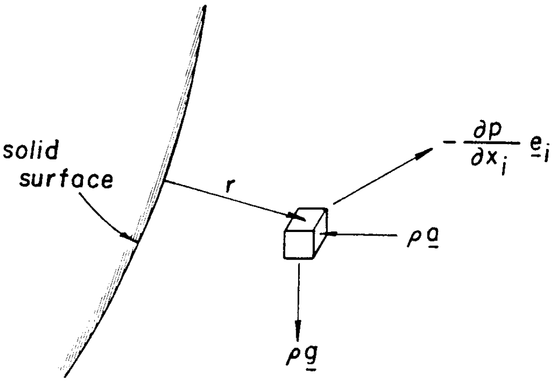

Figure 2.2. A fluid particle in an adsorptive force field.

In Figure 2.2, r is the distance of a fluid particle from a solid surface, **g** is the gravity vector, **a** is the adsorptive force, and $\mathbf{e}_i$ is a unit vector in the direction i. The forces, expressed as force/volume, acting on the particle are indicated. If the potential energy per volume in respect to these force fields are summed and the sum is designated as p*, the result is

$$\mathbf{p}^* = \mathbf{p} + \rho\int_{\mathbf{d}}^{\mathbf{s}} -\mathbf{g}\cdot d\mathbf{s} + \rho\int_{\mathbf{d}}^{\mathbf{s}} -\mathbf{a}\cdot d\mathbf{s}$$

where **s** is a positional vector, **d** represents **s** at a datum point, and ρ is the fluid density treated here as a constant. For practical purposes, the first integral on the right can be represented by ρgh. If the particle does not move within say 100 molecular layers of the solid surface, the second integral is essentially zero in respect to the datum point selected. In this case, the potential energy in respect to the three force fields is given by

$$\mathbf{p}^{*} = \mathbf{p} + \boldsymbol{\rho}\mathbf{gh}$$

and is called piezometric pressure.

If the particle moves into a range where **a** is large, the second integral may take on a large negative value. If this integral is neglected, an error may be made in determining the pressure at points in the pore space. For example, it might be concluded that water is under a large negative pressure (tensile stress), whereas it is only the last integral that is negative. A correct analysis, if one is possible, might show that the water is actually under a substantial positive pressure.

The latter conclusion seems to be justified experimentally because it is known that water in porous media at low saturations has a lowered freezing point, a characteristic of water under positive pressure. On the other hand, the water has a reduced vapor pressure indicating the opposite. This situation can be explained by the fact that adsorptive force fields have a similar effect on the escaping tendency of water molecules as a reduced water pressure.

The foregoing analysis is by no means rigorous since the concept of a fluid particle as previously defined does not apply to fluid within a few hundred molecular layers from a solid surface. Consequently, the analysis must be viewed only as an artifice to aid in the visualization of qualitative effects of adsorptive forces. Furthermore, it is not possible to measure the potential energy resulting from adsorptive forces directly. Instruments for measuring fluid pressures in porous media actually measure the sum of pressure and the potential energy due to adsorptive forces.

Unless one is concerned about problems dealing with freezing point depression, or something similar, this is of no consequence. If the fluid under consideration is under the influence of adsorptive force fields, it is the sum of pressure and adsorptive force that controls the mechanical equilibrium of the fluid. In such cases the measured pressure represents this sum and may be regarded as an "apparent pressure." In the remainder of this text, p is used to designate the measured pressure whether it happens to be the actual pressure or only an apparent pressure.

2.2. DISTRIBUTION OF PRESSURE IN A STATIC TWO-PHASE SYSTEM

Since the resultant of **g** is vertically downward, the force balance can be written as

$$-\frac{dp}{dz} - \rho g = 0$$

in which g is a scalar quantity representing the magnitude of **g**.

The solution of the preceding equation is

$$p - p_o = -\rho g h \tag{2.4}$$

in which h is the elevation above a datum where p is p_o. Equation 2.4 gives the pressure distribution in a static fluid system. When applied to immiscible fluids in a porous medium, it must be applied simultaneously to each of two or more fluids. For example, in the case of a two-phase system

$$p_w - p_{wo} = -\rho_w gh$$

and

$$p_{nw} - p_{nwo} = -\rho_{nw} gh$$

where the subscripts w and nw refer to the wetting and nonwetting phases respectively. Subtracting the first static equation from the second gives

$$p_c = (\rho_w - \rho_{nw}) gh + p_{co} \tag{2.5}$$

in which p_c is the capillary pressure at an elevation h, and p_{co} is the value of p_c at the datum from which h is measured. Provided each phase is interconnected over the interval h, Equation 2.5 gives the value of p_c at all points of contact between phases.

A variation of Equation 2.5 is often used in reference to water in soil profiles under static conditions; that is,

$$p_c = \rho_w gh, \tag{2.6}$$

the density of air being taken as zero, and h being measured above a datum where p_w is at zero gauge pressure. The locus of points where water is at atmospheric pressure is called a water table. In a petroleum reservoir, none of the fluids are at atmospheric pressure, so Equation 2.6 is not applicable. Equation 2.5, however, can be used for any two-phase fluid system in a static condition.

2.3. DEPENDENCE OF SATURATION ON CAPILLARY PRESSURE

Combining Equation 1.5 with Equation 2.5 results in

$$\sigma\left(\frac{1}{r_1}+\frac{1}{r_2}\right) = \Delta(\rho g h) + p_{co} ,$$

indicating that in a static two-phase system, the mean curvature at points on interfaces changes with elevation. Specifically, the radii of curvature are smaller at higher elevations. It is the change in curvature that permits a greater pressure discontinuity across interfaces at higher elevations.

The changes in curvature and capillary pressure are accompanied by a change in fluid distribution. Saturation of the wetting phase is smaller and saturation of the nonwetting phase is greater at greater elevations. A functional relationship between saturation and capillary pressure can be visualized by considering a model of a cross-section of pore space as illustrated in Figure 2.3. In this case, the pore space contains a mixture of water and air.

As capillary pressure is increased from p_{c1} to p_{c2}, either by increasing the air pressure or decreasing the water pressure, a volume of water is removed from the pore space. In the process, the interfaces retreat to portions of the pore space having smaller dimensions, where the radii of curvature are smaller. In other words,

$$S = f(p_c) \cdot$$

However, if the pore space initially is fully occupied with water, a finite value of capillary pressure, designated as p_e, must be exceeded before air can intrude into this element of the pore volume. The value of p_e, called air-entry pressure, depends upon dimensions of the largest opening to the particular element of pore volume considered. If the pore volume

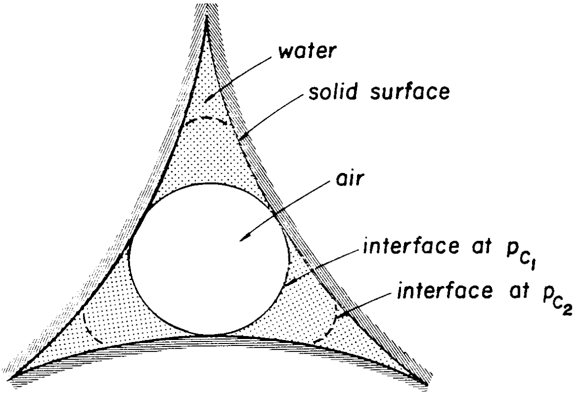

Figure 2.3. Model of pore cross-section with varying saturation of water.

considered contains some portion of pore space with dimensions larger than the largest opening, that portion of the pore space immediately desaturates. Consequently, the desaturation of pore space with increasing capillary pressure does not occur smoothly in a dynamic process but proceeds in jumps that can be observed experimentally [Corey and Brooks (1975)].

The jumps are caused by the entry of air, at ambient pressure, into portions of the medium where no interconnected air previously existed. The pressure of ambient air is suddenly applied to local air-water interfaces resulting in a temporary increase in water pressure until the interfaces can readjust to smaller radii of curvature. The fluctuations have a greater amplitude in media having a wider range of pore sizes. When air is interconnected throughout the pore space further increases in capillary pressure may proceed without noticeable fluctuations.

If a laboratory sample of porous medium is desaturated by increments of capillary pressure and allowed to arrive at a static state with each increment, the values of saturation determined for corresponding values of capillary pressure provide a curve as shown by the solid line in Figure 2.4. In the petroleum literature, it is customary to plot capillary pressure as a function of saturation, as has been done in Figure 2.4 [Richardson (1961)]. Such curves are called capillary pressure-saturation curves [Collins (1961)]. Soil scientists usually plot water content as a function of suction, but the relationship is equivalent, since it is entirely arbitrary as to whether

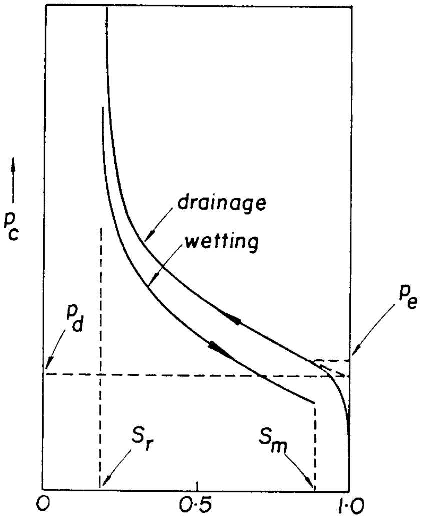

Figure 2.4. Capillary pressure as a function of saturation.

saturation or capillary pressure is regarded as the dependent variable. Soil scientists refer to such curves by a variety of names including water-retention and water-characteristic curves.

2.3.1. Hysteresis

The relationship for $p_c(S)$ depends on the pressure (or saturation) history. In other words, it is subject to hysteresis. A curve, such as the solid line in Figure 2.4, is obtained starting with a wetting phase saturation of 1.0. This is called a desaturation curve in the petroleum literature, and a water-release, drainage, or drying curve by soil scientists.

A different curve is obtained by starting with a sample containing only the nonwetting phase and allowing it to imbibe the wetting phase from a source undergoing successive increases in pressure relative to that of the nonwetting phase. The latter curve is sometimes referred to in petroleum literature as an imbibition curve and in the soils literature as a wetting curve.

The two curves shown in Figure 2.4 belong to an infinite family of curves that might be obtained by starting at any particular saturation and either increasing or decreasing saturation. All such curves, presumably, would be between the two illustrated. Hysteresis has been studied intensively by a substantial number of investigators including Topp (1969).

One factor undoubtedly involved in hysteresis is wettability change, depending upon which phase is first in contact with the solid. Another factor, frequently cited, is illustrated in Figure 2.5 showing two identical capillary tubes with irregular cross-sections. The tube on the left in the figure is first filled and then allowed to drain into the liquid reservoir, whereas the tube on the right was initially empty and then allowed to imbibe liquid from the reservoir. The capillary pressure at the liquid-air interfaces in the tubes is given by Equation 1.5 and the height of rise above the reservoir by Equation 2.6. Since the water stands about twice as high in the tube initially full, the capillary pressure for this tube is about twice as large. The drained tube contains more liquid, because the interface in the tube on the right cannot advance beyond the enlarged cross-section.

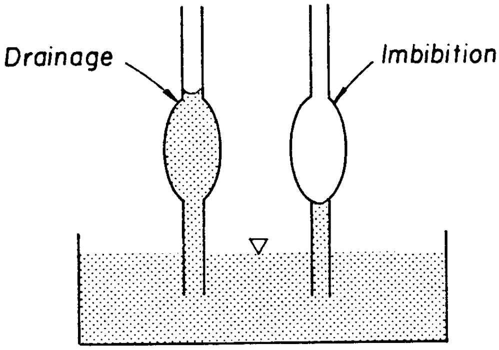

Figure 2.5. Capillary hysteresis.

The situation in ordinary porous media is, of course, more complex than indicated by the simple model, but it is supposed that an analogous mechanism exists in soils, and porous media always contain more water on a drainage than on a wetting cycle for corresponding values of p_c.

Values of $p_c(S)$ are not shown in Figure 2.4 for saturations greater than S_m on the wetting cycle. The reason is that at the critical saturation, the nonwetting phase becomes entrapped. It is no longer interconnected and cannot be replaced simply by decreasing capillary pressure. Since the nonwetting phase is not interconnected, it does not have a unique or a measurable pressure so that capillary pressure cannot be determined. Soil scientists sometimes present data for this range of saturation, but in that case the data represent the negative pressure of water relative to atmospheric pressure, rather than relative to the entrapped air phase.

2.3.2. Entry Pressure

As Figure 2.4 indicates, on a drainage cycle, saturation is very close to 1 over a finite range of capillary pressures. When some critical value of capillary pressure is exceeded, saturation decreases rapidly with increasing capillary pressure. It is supposed [White et al. (1972)] that desaturation (occurring at capillary pressures smaller than that corresponding to the critical saturation) takes place in pore space exposed at the sample boundary. The rationale for this view is that the nonwetting phase cannot reach the interior of a sample until an interconnected network of channels have been desaturated.

It is shown by White et al. that the inflection point on the capillary pressure-saturation curve, where its slope is flattest, corresponds to the highest saturation with an interconnected nonwetting phase. Permeability to the nonwetting fluid is zero at higher saturation.

There is some ambiguity in the literature in regard to what point represents the critical value of capillary pressure on a curve of $p_c(S)$. Petroleum scientists define a critical capillary pressure called displacement pressure p_d as being the capillary pressure at which first desaturation on a drainage cycle occurs. However, since some desaturation takes place in laboratory samples at all finite capillary pressures, it is difficult to decide at what capillary pressure a significant desaturation takes place. Usually, petroleum scientists appear to have defined the displacement pressure by extrapolating the capillary pressure-saturation curve to the ordinate where the saturation is l.0, neglecting that part of the measured data for values of saturations close to 1.0. In the remainder of this text, this is the interpretation assigned to p_d.

It is common practice in the ceramics industry to classify porous ceramics according to the air pressure needed to force air through an initially water-saturated sample. They call this the bubbling pressure p_b. Presumably, this corresponds to the capillary pressure at the inflection point as shown in Figure 2.4. Soil scientists define a similar parameter as the air-entry pressure. Both bubbling pressure and air-entry pressure imply that the nonwetting phase under consideration is air. A term for more general application is simply entry pressure, p_e.

In most cases, the value of saturation at the entry pressure is in the range 0.8 to 0.9 for relatively homogeneous materials, but the entry pressure may vary over a large range, from practically zero to many atmospheres.

2.3.3. Residual Saturation and Effective Saturation

Another parameter of considerable significance is the value of saturation at which capillary pressure increases rapidly with negligible decrease in saturation. This is called the residual or irreducible saturation in the petroleum literature. A somewhat analogous concept in respect to a water saturation found in an otherwise oil-saturated rock is called connate water. Soil scientists sometimes refer to a related parameter θ_{min} as the minimum water content. The term residual saturation designated by S_r is used in this text because the words "irreducible" or "minimum" seem to imply a physical meaning which is not intended.

For example, it clearly is possible to remove practically all wetting fluid from a porous sample by evaporation. Furthermore, plants may remove water from soils at a saturation substantially less than the residual saturation. In many cases, even if the process of removal is restricted to liquid flow, there is no well defined minimum saturation. In spite of its ambiguity, the concept of a residual saturation has practical utility. Corey (1954) and Brooks and Corey (1966) have presented methods of defining residual saturation more or less objectively by extrapolation using data such as are shown in Figure 2.4. Their methods are discussed in Section 2.4. Sometimes residual saturation is determined as the saturation at some arbitrarily large capillary pressure.

The physical interpretation of residual saturation depends to some extent on how it is determined. Some authors have claimed that residual saturation represents wetting phase films over the solid surface or pendular rings (about points of contact of grains) that are not interconnected. This view does not seem to be tenable in view of the observed fact that the wetting phase never becomes completely immobile and the curve of $p_c(S)$ never becomes exactly vertical.

When residual saturation is found by extrapolation, the value determined seems to be associated with a volume of pore space characterized by pore sizes substantially smaller than that of the bulk of the pore space. Many media, porous sandstones being a typical example, frequently have extrapolated residual saturations correlated with the amount of clay in the sample. The residual saturation can be reduced to a very low value, often less than 0.05, by removing the clay with an ultrasonic cleaning technique. Evidently, pore sizes associated with the clay are orders of magnitude smaller than those associated with the remainder of the pore space enclosed by the sand grains.

Furthermore, when the residual saturation is found by extrapolation, a zero value is not uncommon. A zero value of residual saturation often occurs with fine-grained highly-structured soils with a very wide pore-size distribution, even though they usually have a relatively large clay content. Evidently the extrapolated value of residual saturation may be related to a discontinuity in pore sizes such that sizes (associated with $S < S_r$) are distinctly smaller than those corresponding to $S > S_r$. In media with a pore size ranging smoothly from zero to some finite value, the extrapolated residual saturation appears to be zero.

In summary, residual saturation when determined as the saturation at an arbitrarily high capillary pressure, is ambiguous because it depends upon the value of high capillary pressure selected. When residual saturation is determined by an extrapolated procedure, its physical meaning is uncertain, and it probably should be interpreted as a parameter useful for curve-fitting purposes as explained in Section 2.4.

Because the pore space containing the wetting phase at $S < S_r$ contributes relatively little to convective flow, it is convenient (for some purposes) to define an effective saturation S_e given by

$$\mathbf{S_e \equiv \frac{S - S_r}{1 - S_r}} . \qquad 2.7$$

Obviously, S_e is significant mostly in reference to flow, but it is also useful in empirical representations of $p_c(S)$ as is explained in Section 2.4. Similarly an effective or drainable porosity is defined as

$$\boldsymbol{\phi_e \equiv (1 - S_r)\phi} \cdot \qquad 2.8$$

2.3.4. Field Capacity

A concept somewhat related to residual saturation is called field capacity by agronomists and soil scientists (Miller and Klute, 1967). The term is used for two somewhat different concepts. The first is often referred to simply as "field capacity" F.C. and means the water content W in a soil profile after "downward drainage has become very slow" following a thorough wetting. Defined in this way, field capacity may be affected by the position of a water table, if one is present, and by the structure of the entire soil profile as well as the properties of the soil in a particular stratum. Sometimes field capacity is taken as the value of W after an arbitrary period of time following a thorough wetting. A method of measuring field capacity in the field has been described by Peters (1965).

A second usage of the term field capacity may sometimes be called "laboratory field capacity" or the "normal moisture capacity" (Shaw, 1927). In this case, it is determined on a column of soil in a laboratory. A substantial amount of water is added to the top of a column of dry soil, but not enough to wet the soil to the bottom of the column. The entire column is then allowed to reach a pseudo static equilibrium while being protected from evaporation. Values of W are then determined for samples of the soil at elevations sufficiently far from the dry soil to be unaffected by the water content gradient in that region.

Values of laboratory field capacity, when converted to a volume basis by use of Equation 1.1, are often close to values of residual saturation obtained by extrapolation, but sometimes the values are significantly different. Approximations of laboratory field capacity also may be obtained by allowing samples to reach equilibrium at an arbitrarily high suction, for example, in a centrifuge, or in a capillary pressure cell as described in Section 2.5.3.

2.3.5. Measurement of Capillary Pressure as a Function of Saturation

Capillary pressure as a function of saturation apparently was the first functional relationship relating to immiscible fluids in porous media to be measured.

Five methods of determining $p_c(S)$ are known, but there are innumerable variations of each. The methods are:

(1) **Long column -** A long column of porous medium is allowed to reach equilibrium (with a source of wetting fluid at its base) in the Earth's gravitational field. Saturation is determined on samples taken at particular elevations after the column reaches equilibrium. The value of capillary pressure at corresponding elevations is determined from Equation 2.5. According to Collins (1961) this was the first method employed. The long column method is

applicable only for unconsolidated materials which can be packed into a long column. It cannot be used for either undisturbed rocks or soils, and it is useful only for determining $p_c(S)$ at relatively small values of capillary pressure. Furthermore, the time for equilibrium may be very long and uncertain.

(2) **Centrifuge -** A short column of initially saturated porous material (at most, a few centimeters in length) is placed with its long axis horizontal in a centrifuge. The centrifuge is run at some fixed angular velocity until the wetting fluid contained in the sample has reached equilibrium with the centrifugal force imposed by the rotation. A centrifugal force of g x 10^3 is often used. A porous plug is placed at the outer end of the sample during rotation, so that when the rotation is stopped and the sample quickly removed, it is at some arbitrarily high value of capillary pressure. This procedure is usually used to obtain an approximation of residual saturation called the moisture equivalent. An early use of the centrifuge for this purpose was reported by Briggs and McLane (1907).

Another version is to use the centrifuge without the porous plug at the outer boundary of the sample, so that a range of capillary pressure is obtained, from 0 at the outer end to a large value at the inner boundary of the sample. Measurement of saturation at points along the axis of the sample are obtained during rotation using gamma radiation attenuation. An analysis of the forces acting on the fluids and the distribution of fluids along the axis of the sample during rotation are presented in Section 2.5.4.

A centrifuge has been used mostly for obtaining $p_c(S)$ on a drainage cycle. This is because it is difficult to connect the sample to a source of wetting fluid during rotation.

(3) **Vapor pressure -** A sample of porous material with a known wetting phase content is allowed to equilibrate with the atmosphere inside a closed container. The method is limited to cases for which the nonwetting phase is gas, usually water-air systems. The value of capillary pressure at equilibrium is correlated with the vapor pressure of the wetting phase at equilibrium. Versions of the procedure are available for both wetting and drying cycles. The method is useful primarily for studying $p_c(S)$ beyond the range for which capillary pressure is related to pore dimensions by Equation 1.4. This is because vapor pressure is very insensitive to pressure

changes over this range. It becomes more sensitive to changes in liquid saturation, when most of the liquid is under the influence of strong adsorptive forces. In this case, the value of pressure determined is an apparent pressure as explained in Section 2.1.3.

(4) **Pressure cell -** A sample of porous medium is placed in contact with another fully saturated porous medium having an entry pressure such that it will not desaturate at any capillary pressure imposed during the experiment. The porous medium under the sample is called a capillary barrier or a semi-permeable barrier in petroleum literature. The term "semi-permeable" used in this context implies that during the experiment, the barrier (because of its large entry pressure) permits the passage of the wetting phase but not the nonwetting phase.

In the soils literature, a capillary barrier is usually identified by a name alluding to the material from which it is made, e.g., a pressure plate (porous ceramic or fritted glass), or pressure membrane (a thin sheet of porous plastic), etc. In any case, the material must have an entry pressure large enough that it does not permit the breakthrough of the nonwetting phase over the range of capillary pressure needed to desaturate the sample to the degree desired.

A sample is placed on one side of the barrier (usually in a confined chamber surrounded by the nonwetting phase), and the wetting phase is on the opposite side of the container so that its pressure can be controlled. The capillary pressure at the surface of the barrier in contact with the sample is adjusted either by adjusting the wetting-phase pressure in the barrier or by adjusting the nonwetting phase pressure in the sample chamber. When the sample has reached an equilibrium saturation at the imposed capillary pressure its saturation is determined by weighing the sample or by measuring the amount of wetting fluid that was discharged or imbibed during the increment of capillary pressure imposed.

The first use of such a device was reported by Willard Gardner et al. (1922). They called their device a "capillary potentiometer," and used it to obtain an empirical relationship for $p_c(S)$ for a soil on the Utah Agricultural Experiment Station at Logan, Utah. The first plots of $p_c(S)$ analogous to those shown in Figure 2.4 were presented by L.A. Richards (1928). In that paper, Richards described the use of a similar device (now called a tensiometer) for measuring the suction of a soil in the field. He also described plans to determine the conductivity of partially saturated soils using an analogous device, which he eventually succeeded in doing as explained in Chapter 3. A more detailed discussion of the use of tensiometers for measuring the "capillary tension" in soil water was presented by Richards and Gardner in 1936.

Since the early work of Gardner and Richards, the pressure cell has become the most common device for determining $p_c(S)$ by petroleum as well as soil scientists. Innumerable variations of the technique have been used. One variation designed by A. T. Corey was used by White et al. (1970). With this variation, the sample is not enclosed, and evaporation is allowed to proceed continuously. The liquid on the outflow end of the capillary barrier is connected to a capillary tube that contains a liquid-air interface. The tube is connected to a vacuum controller to permit reducing the wetting phase pressure in increments. After each reduction, the wetting liquid moves outward in the tube until evaporation from the sample causes the liquid-air interface to retreat. At the instant the interface begins to retreat, the sample is weighed to determine its saturation. The sample is then returned to the cell, the wetting fluid pressure is reduced by another increment and the process is repeated. The advantage of the latter technique is that a balance of pressure is achieved much more quickly when evaporation is permitted, especially at lower values of saturation.

(5) **Brooks' method -** A method has been devised by R. H. Brooks (1980) which has great utility in respect to obtaining $p_c(S)$ on an imbibition cycle, something difficult to accomplish by most other methods. With the Brooks system, a carefully metered quantity of wetting fluid is added to a porous sample. After a short time (necessary for the wetting-fluid pressure to stabilize) the fluid pressure is recorded using a capillary barrier in contact with the soil and connected to a null-pressure transducer. The saturation is determined from the known quantity of water absorbed.

The advantage of this procedure is that the sample is not required to imbibe fluid from a low pressure source through a capillary barrier. The problem of maintaining excellent contact with the barrier is removed and equilibrium is established in a small fraction of the time required with a pressure cell. A capillary pressure cell based on the latter principle is described by Lorenz, et al. (1992). Their cell is operated by adding or removing a fixed volume of wetting fluid and then closing a valve in the outflow tube. Instead of waiting for the wetting fluid to stop flowing out, or in, the operator waits for the pressure to equilibrate.

2.3.6. Empirical Representations of $p_c(S)$

Brooks and Corey (1966) plotted log S_e as a function of log p_c. They found that if they omitted data for saturations more than about 0.85 the data plotted on a straight line as shown in Figure 2.6, provided that a suitable value of residual saturation is used to compute effective saturation. Brooks and Corey determined residual saturation as the value which provided the "best fit" linear curve. The shape of the curve was found to be sensitive to the value of residual saturation selected. An extrapolation of the linear curve provides a sensitive evaluation of p_d.

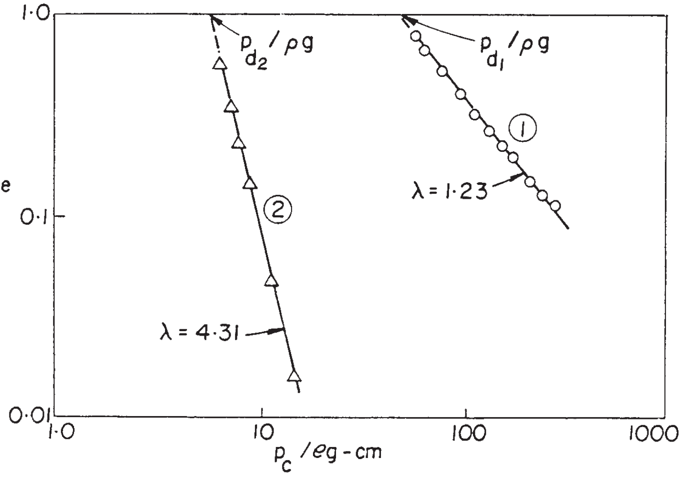

Figure 2.6. Effective saturation as a function of capillary pressure.

As a result of their observation that data for all relatively homogeneous and isotropic samples provided curves as shown in Figure 2.6, Brooks and Corey suggested the empirical relationship

$$S_e \approx \left(\frac{p_d}{p_c}\right)^{\lambda}, \text{ for } p_c > p_d \text{ and } S > S_r \,. \qquad 2.9$$

Originally Equation 2.9 was suggested for use only on a drainage cycle, but Su and Brooks (1975) have applied an analogous relationship (Equation 2.11) for the imbibition cycle as well. In the latter case, a different interpretation is given to the parameter p_d.

Brooks and Corey found that for typical porous media, λ is about 2. Soils with well-developed structure have values of λ less than 2, and sands normally have values of λ greater than 2, sometimes as large as 5 or more.

Equation 2.9 does not represent the measured data for values of $p_c < p_d$. Other empirical expressions have been suggested for this purpose. One such expression, suggested by Laliberte (1969), is

$$\begin{aligned} S_e &= 0.5(1 - \mathrm{erf}\zeta) \qquad S_e \le 0.5 \\ S_e &= 0.5(1 + \mathrm{erf}\zeta) \qquad S_e \ge 0.5 \end{aligned} \tag{2.10}$$

In Equation 2.10, ζ is a function of p_c describing the pore-size distribution.

A semi-empirical expression, suggested by Su and Brooks (1975), is given by

$$p_c = p_e \left[\frac{S - S_r}{a}\right]^{-m} \left[\frac{1 - S}{b}\right]^{bm/a} \tag{2.11}$$

in which a, b and m are constants. The constant m is roughly equivalent to $1/\lambda$ in the Brooks-Corey relationship. Su and Brooks have described the physical interpretation of the constants a and b in their paper. Equation 2.11, like Equation 2.10, is designed to fit data for all values of p_c.

An empirical expression presented by van Genuchten (1980) has gained wide acceptance. This equation, like Equations 2.10 and 2.11, describes $S_e(p_c)$ as a continuous function including values of capillary pressure less than the entry pressure. The equation of van Genuchten is given by

$$S_e = \left[\frac{1}{1 + (\alpha h)^n}\right]^m \tag{2.12}$$

where α, n, and m are constants, and h is the capillary pressure head. The constants n and m are dimensionless, whereas α has the dimensions of 1/L. Although Equation 2.12, as well as Equations 2.11, and 2.12, represent $S_e(p_c)$ as a continuous function, it is possible in each case to choose a set of constants that provide a capillary pressure-saturation curve that approaches the Brooks-Corey relationships closely.

White et al. (1970) have also presented semi-analytical relationships for the entire capillary pressure-saturation curve, but they pointed out that values of $p_c < p_e$ are a function of the external sample geometry and not the pore geometry. According to their theory, the desaturation that occurs at $p_c < p_e$ proceeds only from the boundaries of samples, and consequently is a function of the ratio of external boundary area to volume of the sample.

Occasionally, samples are found with a $p_c(S)$ function unlike that predicted by any of the empirical models. Such samples are often found to be obviously non-homogeneous materials, for example, layered rocks. Curves of log S_e as a function of log p_c often consist of two or more straight-line portions off-set by an increment of capillary pressure and connected by an abrupt transition.

A similar result is produced by samples which consist of a sieve fraction of soil aggregates or a sieve fraction of crushed (but not pulverized) sandstone rock. In this case, the two-staged curve (sometimes called bimodal) undoubtedly results from a discontinuity in pore size between the primary and secondary pore space.

2.4. PORE-SIZE DISTRIBUTION

From a force balance similar to that used in Section 1.5.1 to obtain Equation 1.4, it is possible to obtain capillary pressure for an interface across an irregular shaped space such as may exist in soil pores. In general, the line of contact of an interface with the internal solid surfaces of porous media is not circular. However, because of the tendency of interfaces to reach a minimum area consistent with the saturation, the line of contact tends to lie nearly in a plane.

In the following analysis, sections of interfaces are considered which are concave toward the nonwetting fluid and which possess orthogonal radii of curvature of the same sign, as well as having lines of contact that lie more or less in a plane. Adsorbed films and pendular rings are excluded. A sketch of a portion of pore space (across which several such sections might be positioned) is illustrated in Figure 2.7.

A balance of forces across a particular section indicates that

$$\bar{p}_c A \approx \sigma \cos\alpha \ wp$$

where wp designates the wetted perimeter, that is, the length of the line of contact. The area A represents the area of the plane passing through wp enclosed by wp. The force balance presumably would be exact if the wp

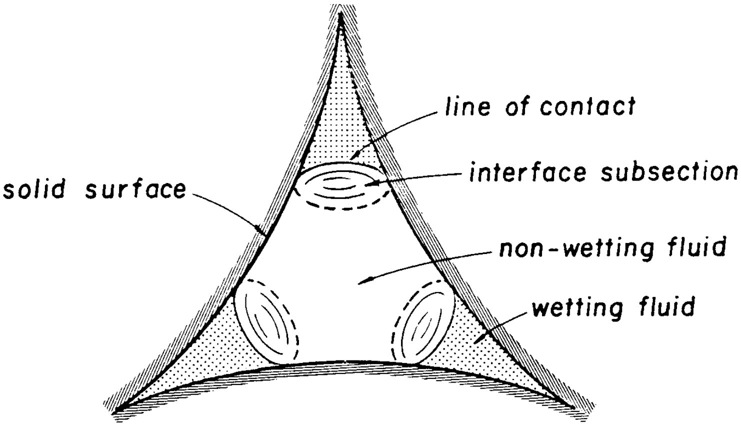

Figure 2.7. Cross section of pore space with several interface sections.

under consideration actually lies exactly in one plane, if the contact angle α is constant over wp, and if σ is constant over the section.

Rearranging gives

$$\frac{A}{wp} \approx \frac{\sigma \cos \alpha}{\overline{p}_c}. \tag{2.13}$$

The quantity A/wp has the dimension of length and can be used to characterize the size of the section of pore space across which interfaces are positioned. Note, that this concept is consistent with the definition of pore size given in Section 1.3.3. It is also consistent with the concept of hydraulic radius. However, the geometry of a particular pore region might not be characterized by the value of capillary pressure in surrounding regions if the nonwetting phase is blocked from entering the region by smaller pore sizes at all points on the boundary of the region.

The following analysis assumes that practically all portions of the pore space with pore sizes larger than that given by Equation 2.13 have access to interconnected nonwetting fluid. This may appear to be a severe restriction, but evidence presented by White et al. (1972), and Corey and Brooks (1975) imply that the condition is satisfied for practical purposes at saturations smaller than that corresponding to the inflection point in the liquid retention curve occurring at the entry pressure. However, pore sizes represented by $p_c < p_e$ must be characterized by extrapolation.

A volume element of the porous medium is considered that is large enough to be representative of a particular sample but small enough that capillary pressure can be represented by a single value when the fluids are in a static condition on the drainage cycle. This implies that the elevation difference within the element is negligible. Within such an element, there usually are many sections of the interface. The quantity A/wp should be the same for each, regardless of the orientation of A. Furthermore, if the medium is isotropic, the orientations of A for each section is assumed to be random, that is, there is no preferred orientation.

If capillary pressure is increased within the reference element, the value of A/wp decreases. In the process, a portion of the pore space loses its wetting phase. The increment of saturation is a measure of the fraction of pore space characterized by the corresponding increment of A/wp. The ratio of change in saturation to change in A/wp is dependent upon the frequency of pore sizes having values of A/wp within the increment of capillary pressure considered.

Equation 2.13 indicates that if $\sigma \cos\alpha$ is essentially constant for a particular fluid system, A/wp is proportional to $1/p_c$. The quantity $1/p_c$, therefore, should be a measure of the largest pore size (in that part of the pore space containing the wetting phase) at a particular value of capillary pressure. Similarly, a plot of saturation as a function of p_d/p_c should provide an indication of the distribution of sizes characterizing the pore space. Figure 2.8 presents contrasting examples of this kind of plot.

In Figure 2.8, S_e rather than S has been plotted as a function of p_d/p_c. By normalizing the ordinate and abscissa in this way, data for any medium can be plotted on the same graph without changing the scale. Furthermore, for media with similar pore-size distributions, the data plot on the same curve. In any case, the end points of all curves are the same, that is, 0 and 1.0. This is because both S_r and p_d are chosen by an extrapolation which forces the curves to behave in this way. It should be observed that p_d is somewhat smaller than p_e.

Data for $p_c < p_e$ are disregarded in the extrapolation process. The rationale for this is that, for $p_c < p_e$, some regions of the pore space may be isolated from an interconnected nonwetting phase. The geometry of isolated pore space is not characterized by $p_c(S)$ data for $p_c < p_e$. Pore sizes larger

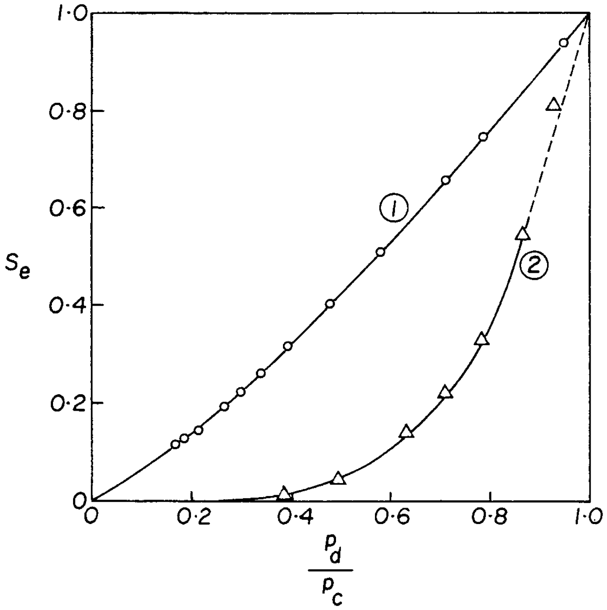

Figure 2.8. Effective saturation as a function of p_d/p_c

than that corresponding to the entry pressure are assumed to be determined by the extrapolation process, the largest being characterized by p_d.

Data for high values of capillary pressure where the saturation approaches residual are also disregarded. The reason for this is that a significant portion of liquid removed at such saturations is associated with surface films and pendular rings. There is no reason to suppose that Equation 2.13 is valid in this situation, or that the pore geometry is described by $p_c(S)$ where S approaches S_r.

Curves of $p_c(S)$ have been used by many investigators to characterize pore-size distribution in porous media. Perhaps the first to suggest this were Childs and Collis-George (1948) and Purcell (1949). However, none of the investigators employing this principle have pointed out the apparent fallacy of including data for $p_c < p_e$ in the characterization of pore geometry.

Many investigators, Purcell being among the first, have used data for $p_c(S)$ obtained by a mercury injection procedure. With this procedure, mercury is injected as a nonwetting phase into a porous sample that has been first evacuated. The value of capillary pressure is taken as the pressure required to inject a given increment of mercury into the pore space, and saturation is calculated from the volume of fluid injected. The method of mercury injection has the advantage of speed, but it does not evaluate any changes in pore size that might result from the reaction of porous media to fluids of the prototype system.

The first step in plotting curves of the type shown in Figure 2.8, after the data are measured, is to determine values of S_r and p_d. One way of accomplishing this is as follows:

(1) The data for $p_c(S)$ are plotted as shown in Figure 2.4.

(2) Estimations of p_e, S_r, and p_d are made, and the estimated values are designated as p_e', S_r', and p_d'.

(3) Data representing $p_c < p_e'$ and $S < S_r'$ are disregarded.

(4) Using the remainder of the data, a plot of $(p_d'/p_c)^2$ as a function of S is made as shown in Figure 2.9. For typical pore-size distributions, the plot is roughly linear. This is because an average value of λ is about 2 [see Equation 2.9].

(5) The plot is extrapolated to the abscissa to obtain S_r.

(6) The plot is extrapolated to the ordinate to obtain the intercept I.

(7) p_d is calculated from

$$\mathbf{p_d = p_d' / \sqrt{I}} \cdot$$

(8) The improved values of S_r and p_d are used to calculate S_e and corresponding values of (p_d/p_c) to plot curves of the type shown in Figure 2.8.

More sensitive extrapolation procedures for obtaining S_r and p_d can be devised, such as the method presented in Section 2.3.6. A variety of extrapolation methods using a computer have been devised that provide more or less the same results.

Another method of illustrating the size-distribution of pore space is by plotting $dS_e/d(p_d/p_c)$ as a function of p_d/p_c. Examples of such plots are

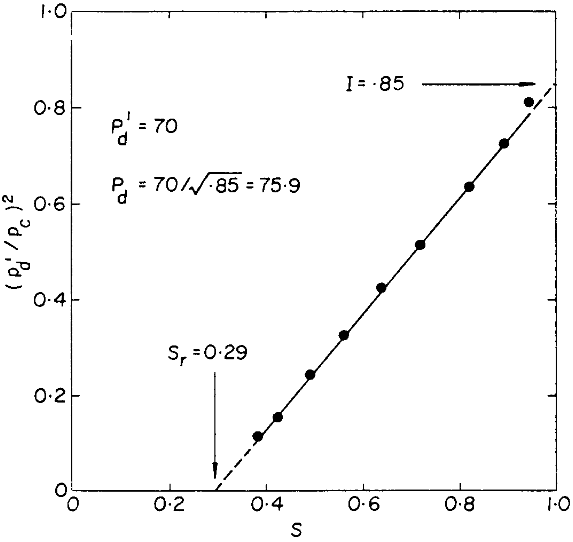

Figure 2.9. Graphical procedure for estimating S_r and p_d.

shown in Figure 2.10. Curves 1 and 2 represent the same measured data as curves 1 and 2 in Figure 2.8.

The plots shown in Figure 2.10 can be regarded as frequency distribution curves because they indicate the rate of change of S with respect to a function proportional to pore size. The area under such curves is 1.0.

A medium having a wider range of pore size, e.g., Sample 1, has a curve less sharply peaked and its area is spread over a wider range of values than is the case for Sample 2. According to White et al. (1972), the highest value of the ordinate occurs at the entry pressure. Undoubtedly, this is because an interconnected nonwetting phase cannot reach all regions of a medium until the entry pressure is reached. For this reason, parts of the curves representing $p_c < p_e$ are shown with dotted lines.

2.4.1. Pore-size Distribution Index

Brooks and Corey used the parameter λ, as shown in Figure 2.6 and Equation 2.9, as an index of pore-size distribution. They reasoned that for

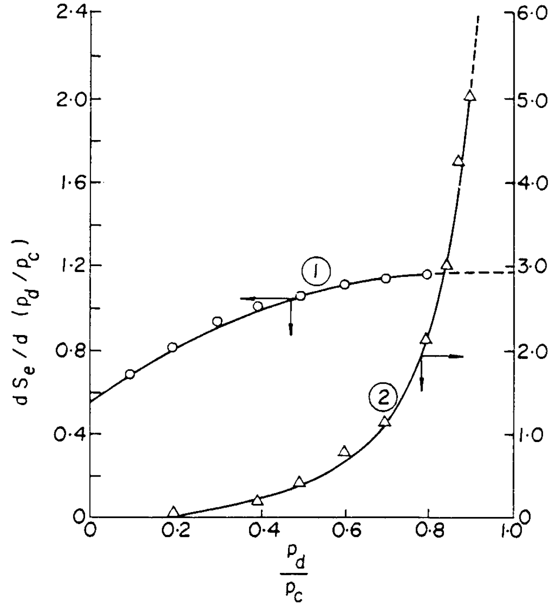

Figure 2.10. Frequency distribution of pore-size as a function of p_d/p_c.

media having a uniform pore-size, the index would be a large number that theoretically could approach infinity. On the other hand, media with a very wide range of pore sizes should have a small value of λ that theoretically could approach zero. They found that typical porous media, have values of λ about 2.

In the case of naturally occurring sand deposits, λ is often about 5 or 6, especially if the material is thoroughly mixed and densely packed. For soils in an undisturbed state $\lambda < 1$ is not uncommon.

2.4.2. Factors Affecting Pore-size Distributions

The effect of grain-size distribution, structure and mixing on pore-size distribution is mentioned in Section 1.3.4. The grain-size distribution is probably the least important. There is a tendency, however, for finer materials to have smaller values of λ. All sands, regardless of grain-size distribution can be made to have a very uniform pore-size by thorough mixing and dense packing. The mixing and packing can be accomplished by dropping the sand through at least two screens a few centimeters apart as suggested by Wygal (1963). Even undisturbed sands found in natural deposits usually have relatively high values of λ, say 4 or 5.

Laliberte and Brooks (1967) experimentally determined the relationship between porosity and λ for several soils and sands, and found that λ increases as porosity decreases. Evidently, the decrease in porosity reduces the range of pore sizes as expected. Structure increases the range of pore sizes and decreases λ. Shape of sand grains theoretically could also have an effect, but this possibility has not been systematically investigated to date. It is not known, for example, whether a freshly crushed deposit of rock fragments would have a significantly smaller value of λ than sand from a stream bed. The degree and type of cementation are other possible factors that have not been studied systematically.

2.5. DISTRIBUTION OF FLUIDS IN STATIC SYSTEMS

If the pressure history for a static system is known and the relationship $p_c(S)$ can be measured, for each distinct medium contained in the system, it is possible to deduce the distribution of fluids. This is accomplished by an application of Equation 2.5. Examples of static fluid distributions are described, some relating to field systems and some relating to laboratory devices, in the following sections.

2.5.1. Soil-water System

As a first example of the application of Equation 2.5, a soil profile, thoroughly soaked following a rain or irrigation, is considered. If such a soil is protected from evaporation and from large temperature fluctuations, the soil-water system eventually approaches equilibrium. If there is a water table at some depth below the surface, the fluid system approaches equilibrium with the water table and the water table then can be used as a datum from which to measure the elevation and pressure.

In the case of a soil in the field, a pseudo static state can be expected only in a fallow soil, where plants are not present to remove the water continuously and where enough surface soil has dried to protect the remainder from significant evaporation. In such cases, the water below the dry layer may approach a static distribution.

Field soils never reach a condition of complete saturation, either above or immediately below the water table. This is because some entrapped gases are always present. Usually about 8-20 percent of the pore space is occupied by gases, even where the water saturation is a maximum. Under such circumstances a saturation distribution similar to that shown in Figure 2.11 is expected if the soil profile is homogeneous.

In Figure 2.11, h represents an elevation measured above the water table, that is, the locus of points at which the water is at atmospheric pressure. The water content both immediately above, as well as below, the water table is at some maximum value determined by the saturation at which gas ceases to be interconnected on the wetting cycle following rains or irrigations.

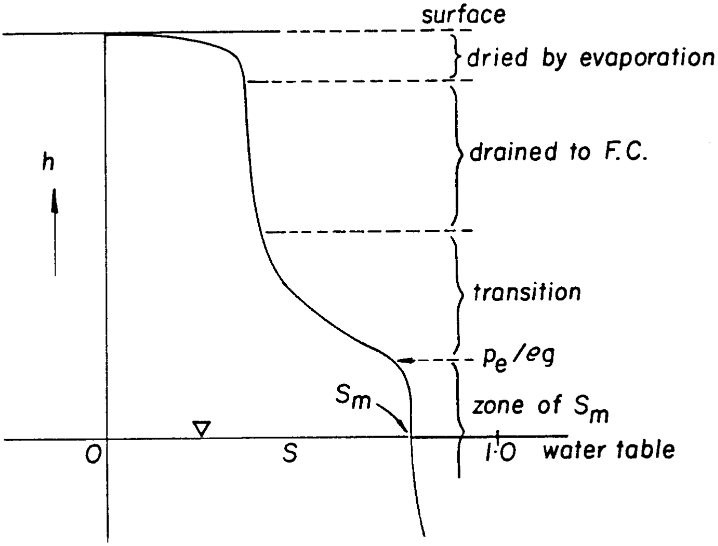

Figure 2.11. Distribution of pseudo static water in a homogeneous soil profile with water table.

Part of the water distribution curve for the profile (below the zone of evaporation and above the water table) is similar to the relationship shown in Figure 2.4 except that the ordinate p_c has been replaced by h in conformance with Equation 2.6.

A second example of a pseudo-static water distribution is shown in Figure 2.12. In this case there is no water table, a layer of dry soil being beneath as well as above the moist soil.

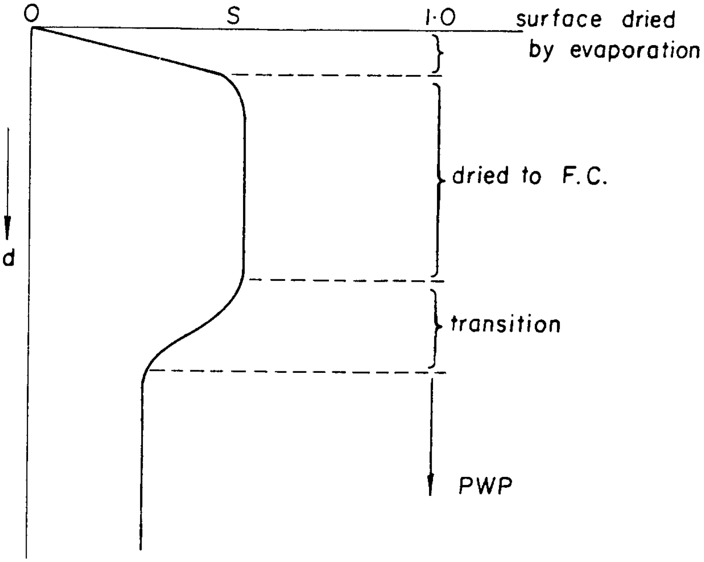

Figure 2.12. Distribution of pseudo static water in a homogeneous soil without a water table.

The abbreviation PWP in Figure 2.12 refers to the permanent wilting point, a term used by soil scientists to designate the water content remaining when plants growing in the soil permanently wilt. PWP usually is about half of FC but is greater than the water content of air-dried soil. The distributions illustrated in Figures 2.11 and 2.12 do not represent conditions of true static equilibrium because the presence of sharp transitions from wet to dry soil imply a pressure gradient causing some flow. The enormous resistance in regions of dry soil results in a pseudo static condition in the wet soil. This situation is more easily explained by theory discussed in Chapter 4.

Soil profiles often consist of layers having contrasting properties. To determine a water distribution curve for such a case, it is necessary to obtain a $p_c(S)$ relationship on the appropriate cycle for each layer. The distribution of capillary pressure with elevation when a water table exists is not affected by the presence of contrasting strata, provided the system is truly static. However, even an infinitesimal flow may change the distribution of capillary pressure drastically. Consequently, the distribution of fluids in non-homogeneous media is better discussed in the context of theory presented in Chapter 4.

A question relative to a soil-water system as illustrated in Figure 2.11, is how much water drains from the soil when the water table is lowered to another static position. In some literature, the volume of water per unit surface area that drains when the water table is lowered by a unit increment is called specific yield. In this text specific yield S_y is defined by

$$S_y \equiv \frac{dV_d}{dD} \qquad 2.14$$

in which V_d is the volume drained per unit surface area and D is the depth to the water table from the dry surface layer.

Equation 2.9 provides a convenient tool for evaluating the derivative in Equation 2.14. In laboratory samples, Equation 2.9 is not valid for $p_c < p_e$, but the boundary effect causing the initial desaturation is not expected to have a significant effect in the soil profile. This is because air can replace water only by moving downward through the desaturated zone and not from lateral boundaries as would be the case with small samples.

For the field soil, however, the value of S is not 1.0 at any point, even below the water table where air is trapped. Therefore, S_e is redefined for this case as

$$S_e \equiv \frac{S - S_r}{S_m - S_r}$$

in which S_m is the maximum field saturation. Also, Equation 2.9 is modified in terms of Equation 2.5 as

$$S_e \equiv \left(\frac{h_d}{h}\right)^{\lambda}$$

for $h > h_d$, in which h_d corresponds to an elevation at which p_c is p_d. The meaning of p_d may be modified slightly because of the fact that the maximum water saturation is S_m rather than 1.0.

The volume drained below the dry layer is given by

$$V_d = \phi_e \int_0^D \left(1 - S_e\right) dh$$

in which ϕ_e is redefined as

$$\phi_e \equiv \phi\left(S_m - S_r\right).$$

Expressing S_e in terms of h gives

$$V_d = \phi_e \int_{h_d}^{D} \left[1 - \left(\frac{h_d}{D}\right)^{\lambda}\right] dh \; .$$

for $h_d \leq D$. Since the derivative of an integral with respect to a variable upper limit (the lower limit being constant) is the integrand evaluated at the upper limit,

$$S_y = \phi_e \left[1 - \left(\frac{h_d}{D}\right)^{\lambda}\right] . \qquad 2.15$$

for $D < h_d$,

$$S_y = 0 \; .$$

For $D >> h_d$,

$$S_y \approx \phi_e \, .$$

It is often assumed by groundwater and drainage engineers that specific yield is equivalent to the effective or drainable porosity. However, Equation 2.15 shows that this is a valid assumption only when the second term in the brackets on the right side of Equation 2.15 is negligible. For sandy soils with λ about 4 or 5, effective porosity is a good approximation of the specific yield when the depth to the water table is only about 2 h_d. However, for structured soils with pore-size indices less than 1.0, the specific yield is significantly less than the drainable porosity unless the water table is at a depth much greater than 2 h_d.

It often happens, especially in respect to drainage design, that engineers make a larger error than is realized by regarding specific yield as a constant.

This fact has been pointed out by Duke (1972) who is responsible for the derivation of Equation 2.15. An excellent discussion of the concept of specific yield has been presented also by Bear (1972).

2.5.2. Petroleum Reservoir

An oil-bearing rock formation is another example of a system which sometimes may contain practically static fluids. In such cases, the system has fluid distributions somewhat analogous to those of a water-air system in a soil.

A typical petroleum reservoir is a stratum of porous rock confined by relatively impermeable rock. Usually the petroleum fluids, oil and gas, occupy only a small fraction of the porous stratum, and even there, they are mixed with brine saturating the remaining and larger portion of the stratum. If the fluids are static, the oil and gas are in the higher part of the stratum, with gas at the apex.

Fluid carbon compounds, which make up the oil and gas, were formed in mud buried in saline marine deposits with a shortage of oxygen. Later the mud underwent metamorphosis and became shale. In this process, the heat and pressure forced much of the oil and gas into adjacent aquifers, where it could accumulate and become mobile.

The petroleum fluids were able to accumulate, however, only in formations that were sealed. Such formations were necessarily confined between adjacent impermeable strata, and capped at the highest point. Owing to their buoyancy, the density of most petroleum fluids being less than that of the brine originally fully saturating the aquifer, the petroleum fluids tended to migrate upward until blocked by the cap rock.

Often a petroleum body is found in a formation at the apex of an anticline, as illustrated in Figure 2.13, or where the oil-bearing aquifer slopes upward to a point where it is blocked by a fault. In some cases, discussed in Chapter 4, the cap rock is also porous but has an entry pressure (oil into brine) sufficient to prevent further upward migration. Often, the cap rock is sufficient to provide a significant accumulation only under dynamic conditions in which brine is flowing down slope.

Fluids in petroleum reservoirs are usually at elevated temperatures and at pressures much greater than atmospheric. Consequently, there is nothing completely analogous to a water table in a petroleum reservoir. Also, the carbon compounds often exist in two phases, i.e., liquid and gas. The gas occupies the apex of the petroleum body as shown in Figure 2.13.

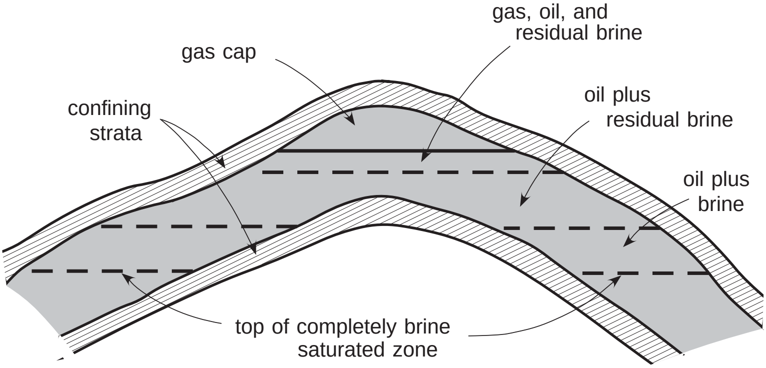

Figure 2.13. Schematic of petroleum reservoir with static fluids.

Among the first questions to be considered is how much oil is recoverable, and where is the best place to drill wells to accomplish the most efficient recovery. As the oil is removed, the brine migrates upward and the gas migrates downward until eventually oil production becomes unprofitable, either because too much brine is produced along with too little oil, or the gas breaks through to the wells. After gas breakthrough, the reservoir pressure may fall precipitously, dissipating the energy available for recovering oil.

The distribution of fluids in a virgin reservoir with static fluids is estimated by first calculating the distribution of capillary pressure with elevation. The lower edge of the hydrocarbon region may be used as a starting point for the calculation of capillary pressure. The data needed are:

(1) the densities of each of the fluid phases, including brine,

(2) p_e for oil into brine,

(3) p_e for gas into oil,

(4) the elevations of the lower edges of the oil and gas bodies, or,

(5) the absolute pressures of the oil, gas and brine at some point or points where these three phases exist,

(6) S_w, S_o and S_g as functions of p_c for oil into brine and gas into oil.

The value of capillary pressure at the brine-oil interfaces is given by Equation 2.5 where h is the elevation above the bottom of the oil body and p_{co} is p_e (oil into brine) for the rock at that position. Similarly, p_c (gas into oil) in the upper portion of the stratum is given by Equation 2.5 where h is the elevation above the bottom of the gas body and p_{co} is p_e (gas into oil) for the rock at that position. The brine, oil and gas distributions can be estimated from $p_c(S)$ curves although a complication exists within that portion of the gas cap where three fluid phases may exist. This is usually a relatively narrow zone, however.

A possible distribution of fluids existing in such a reservoir is shown in Figure 2.14, in which the distribution is with respect to elevation above the bottom of the oil body.

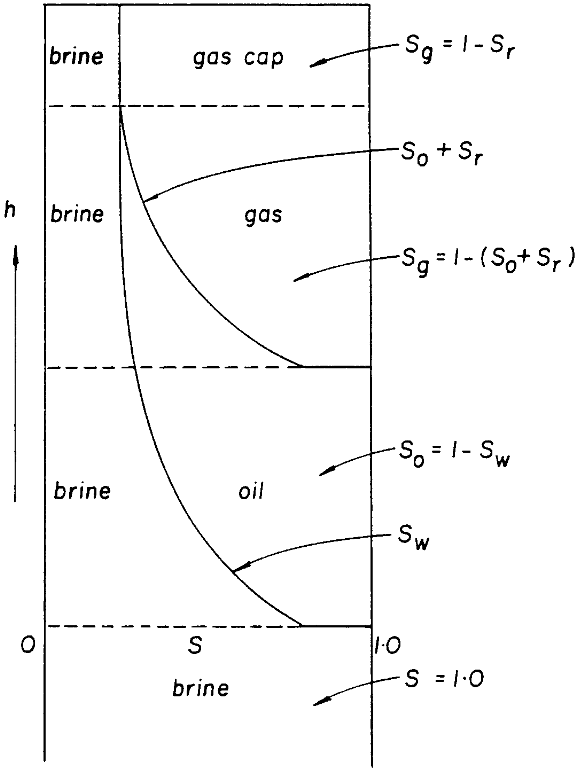

Figure 2.14. Distribution of fluids in a petroleum reservoir.

Complications are caused by stratification as in the case of soils. Furthermore, a stratum of oil in a state of true equilibrium is rarely found even for virgin reservoirs. Often the brine is found to be flowing slowly in one direction or another, and this has a large effect on the distribution of fluids. In the static case, however, brine saturation decreases and oil saturation increases with elevation until the brine saturation is residual. At some elevation, a gas body begins, at which level p_c (gas into oil) is p_e for gas into oil. Above this level, the oil saturation decreases and brine remains at the residual saturation. At a higher level, the oil body may disappear completely so that only gas and residual brine remain. The reservoir portion containing only gas and residual brine is called a gas cap.

2.5.3. Fluids in a Porous Solid in Equilibrium with a Capillary Barrier

A sketch of a typical capillary-pressure cell is shown in Figure 2.15. As explained in Section 2.3.5, a pressure cell is used to control capillary pressure at one boundary of a saturated porous medium, that is, at the boundary of a capillary barrier. When a porous sample is placed in contact with the barrier, its saturation adjusts until the capillary pressure of the sample at the point of contact reaches the controlled value.

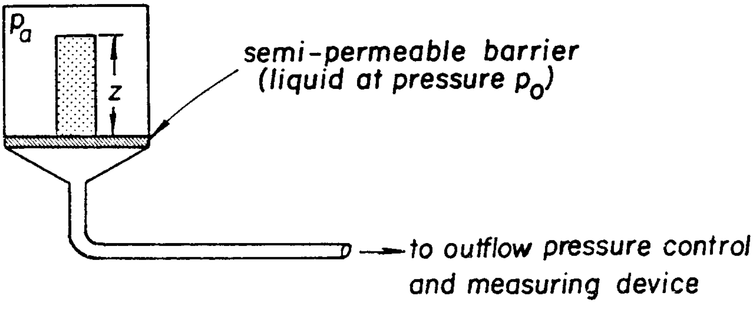

Figure 2.15. Capillary-pressure cell.

In reference to Figure 2.15, if p_o is maintained in the liquid at the top of the barrier and a pressure p_a in the gas within the chamber, the following relationships hold at equilibrium:

At z = 0

$$p_c = p_a - p_o \,.$$

At any z,

$$p_c = p_a - p_o + \rho_w g z.$$

At $p_o = 0$ and $z = 0$,

$$p_c = p_a \,.$$

It is necessary that the temperature and vapor pressure inside the sealed chamber also be allowed to reach equilibrium. For example, if the chamber is connected to an air pressure regulator, provision should be made to minimize diffusion of the wetting-fluid vapor through the air line to the regulator.

2.5.4. Fluids at Equilibrium in a Porous Solid in a Centrifuge

A small sample of porous material (containing a wetting liquid) rotated about an axis in a centrifuge is sketched in Figure 2.16.

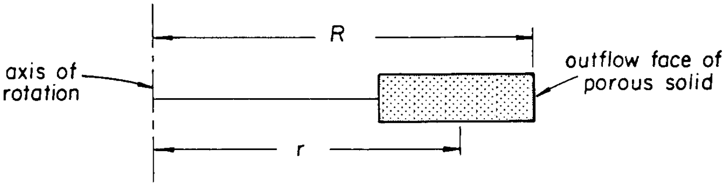

Figure 2.16. Schematic of porous sample in centrifuge.

The condition of the liquid in the sample (after the centrifuge has been rotated for a sufficient time at a constant angular velocity ω) can be analyzed as a problem of statics. The fluid elements no longer move with respect to the solid matrix. At this time, the pressure gradient in the liquid is such as to balance the centrifugal force and gravity. In this case gravity can be ignored because centrifugal force with a magnitude of about g x 10^3 is imposed.

Since the centrifugal force is in the outward direction on fluid elements, the negative pressure gradient must act inward. The balance of forces is given by

$$-\frac{dp_w}{dr} + \rho\omega^2 r = 0.$$

where r is a distance outward from the center of rotation, and ω is the angular velocity . If the density and pressure of air are taken to be zero,

$$dp_c = -\omega^2 \rho r\, dr + dr.$$

Integrating gives

$$p_c = -\frac{\omega^2}{2} \rho r^2 + c.$$

A consideration of the physics of this system indicates that on the outflow face of the sample, at r = R, the liquid pressure is only negligibly greater than the pressure of surrounding air. Thus

$$c \approx \frac{\omega^2}{2} \rho R^2$$

and

$$p_c \approx \frac{\omega^2}{2} \rho \left(R^2 - r^2\right). \qquad 2.16$$

where R is the radius at the outflow end of the sample.

Capillary pressure varies from zero where the radial distance is R to a higher capillary pressure towards the center of rotation, depending upon the rate of angular rotation. In practice capillary pressure can be made very large at a short distance from the outflow face. If it is desired to determine the residual saturation for a sample, a porous plug is placed in contact with the outflow face of the sample. This, in effect, moves the outflow face of the system outward from the sample and places the entire sample in the high capillary pressure region. When the rotation of the centrifuge is stopped, negligible redistribution of water in the sample takes place, because the entire sample is at residual saturation and has a negligible water conductivity.

PROBLEMS AND STUDY QUESTIONS

1. Consider a sealed tube of soil partially saturated with water. A small temperature differential is maintained continuously across the ends of the tube. (a) Will the tube eventually reach a condition of thermal equilibrium? (b) Will it reach a condition of mechanical equilibrium? Give reasons for each answer.

2. Consider the same situation as described in question 1, except that the soil is replaced by a hypothetical medium having pores small enough to prevent bulk flow (too small to hold fluid elements). Answer question (a) and (b) above and give reasons.

3. Explain why pressure p is treated as a scalar quantity, whereas the surface stress, $\boldsymbol{\sigma}$, is treated as a vector.

4. Explain why (when balancing forces on a reference element of fluid) it is not necessary to consider surface forces produced at solid-fluid boundaries, whereas for macroscopic reference elements, this is necessary.

5. Give a possible reason why soil water at a negative gauge pressure of 5 atmospheres does not cavitate, whereas water in pipes usually cavitates at a negative pressure of 1 atmosphere.

6. What is the meaning, if any, of capillary pressure in a soil at a point a few centimeters above a water table? How could this capillary pressure be measured?

7. Consider two capillary tubes, one having a circular cross section, and the other a square cross section. If both tubes have the same cross-sectional area, which tube is likely to have the higher capillary rise? Explain. Answer the same question in respect to a capillary tube having a rectangular cross section, with one side being twice the dimension of the other.

8. Consider a straight capillary tube with uniform cross-sectional area but with varying cross-sectional shape. Would such a tube be expected to exhibit hysteresis in an analogous way to the tubes illustrated in Figure 2.5? Explain.

9. It is supposed that the small amount of desaturation occurring at capillary pressures less than the entry pressure is due to retreating interfaces at the sample boundary. Describe a possible experiment to provide evidence of this premise.

10. Would the correlation between residual saturation (found by extrapolation) and field capacity be better for a sandy soil or a fine-textured soil with a well-developed structure?

11. If a porous sample 5 cm in length is placed in a centrifuge to determine capillary pressure-desaturation curves, what length of sample would have to be employed by the "long-column" method to obtain the same range of capillary pressures? Assume the centrifuge is operated at the usual angular velocity.

12. Would you expect the long-column procedure for determining capillary pressure curves to be popular in petroleum industry for characterizing porous rocks? Explain.

13. Would you expect a piezometer to be equally as effective as a tensiometer for measuring the pressure of water in soil within the root zone of crops? Explain.

14. Why would one want to determine the pressure of soil water within the root zone of crops?

15. Would you expect the value of λ in the Brooks-Corey equation to be larger for a medium consisting of freshly crushed quartz particles, or for sand taken from a swift flowing river bed? Assume that both media have the same grain-size distribution.

16. Other things being equal, would you expect a packing technique that results in a smaller porosity to result in a larger or smaller value of λ for a given granular sedimentary material in a laboratory column?

17. If the entry pressure head measured with air into water is 50 cm for a particular sandstone, what would you expect entry pressure (expressed in atmospheres) to be for mercury into air?

18. Consider a soil containing static water, with a water table at a fixed elevation beneath the surface. If the soil texture varies in horizontal planes, describe qualitatively how you would expect the saturation to vary in horizontal planes in respect to the entry pressure.

19. Explain why the term "pseudo equilibrium" is used to describe the condition of water in fallow soils.

20. It is desired to locate the lower edge of an oil body and also the lower edge of a gas body in a static petroleum reservoir. Assuming that it is possible to measure the pressure of brine, oil and gas independently at the bottom of an observation well, explain where such a well might be drilled to obtain the necessary data from a single hole. Describe all the data needed and explain how it would be used to make the necessary calculations.

21. The Brooks method (of determining capillary pressure-saturation curves on the wetting cycle) reduces the importance of excellent contact between sample and capillary barrier as compared to the pressure-cell procedure. Explain.

22. When using a centrifuge to obtain the moisture equivalent of a sample of porous rock, it is important to obtain good contact between the sample and a porous plug at the outer face of the sample. Explain.

23. Sketch a hypothetical capillary pressure-saturation curve such that the specific yield would not be a function of depth to the water table. What would the displacement pressure be for this case?

24. The U.S. Bureau of Reclamation assumes a constant value of specific yield in their procedure for the design of drainage systems. List all of the conditions that are necessary for this to be a valid approximation.

25. Would you expect that the use of mercury injection (to obtain capillary pressure-saturation curves for characterizing the pore-size distribution) to be more valid for a sandstone containing negligible clay or for one that contained considerable montmorillonite? Explain.

Chapter 3

EQUATIONS OF FLUID FLUX IN POROUS MEDIA

3.1. FLUID MOTION

According to Newton's second law, the rate of change of momentum with respect to time (of an element of mass) is equal to the resultant force acting on the element. If the mass of the element does not change with time, Newton's law implies that the product of mass and acceleration is equal to the force. In the following analyses, Newton's law is written in respect to a fluid particle, as defined in Section 1.4.2. It is assumed that the particle undergoes negligible divergence.

In this case, the only force acting on the particle, other than the driving forces that act also on static elements, is fluid shear. With this simplification, Newton's law indicates that, for any direction i,

$$\rho \frac{du_i}{dt} = \rho g_i - \frac{\partial p}{\partial x_i} + F_i\,(\text{shear}). \tag{3.1}$$

In Equation 3.1, all force terms are expressed on a per volume basis. The left side represents the product of mass and acceleration, and the right side is a summation of force components in the i direction.

The first two terms on the right represent the driving forces as described in Section 2.1.1. The terms representing acceleration and shear force components, however, require further explanation.

3.1.1. Fluid Velocity

The component of velocity u_i refers to motion of the center of mass of a fluid particle. It does not refer to the motion of individual molecules or ions, or to any species of molecules that constitute the fluid mass. It is required, for the analysis presented here, that the particle contains the same mass at all times as it moves in space, but the individual molecules contained in the element are continuously interchanging with those of neighboring elements. Consequently, the fluid particle is an abstraction rather than a physical entity.

It is assumed that the derivatives of u_i (of any order) with respect to the space coordinates exist. Thus, u_i is regarded as an analytic function of the space coordinates throughout a fluid phase including the solid boundaries. This assumption has limitations as explained in Section 3.7.2.

3.1.2. Fluid Acceleration

The derivative du_i/dt is the "total" component of acceleration in the direction i. Noting that, in general, u_i is a function of orthogonal space coordinates, x_i, x_j, and x_k , as well as t,

$$du_i = \frac{\partial u_i}{\partial x_i} dx_i + \frac{\partial u_i}{\partial x_j} dx_j + \frac{\partial u_i}{\partial x_k} dx_k + \frac{\partial u_i}{\partial t} dt.$$

Dividing by dt gives

$$\frac{du_i}{dt} = \left(u_i \frac{\partial u_i}{\partial x_i} + u_j \frac{\partial u_i}{\partial x_j} + u_k \frac{\partial u_i}{\partial x_k} \right) + \frac{\partial u_i}{\partial t}$$

in which u_i, u_j, u_k are velocity components in the x_i, x_j, and x_k directions respectively.

Writing this with the summation convention results in

$$\frac{du_i}{dt} = u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial t}, \qquad 3.2$$

the repeated subscript j indicating a summation over three orthogonal coordinate directions.

The first term on the right of Equation 3.2 is the convective acceleration due to velocity variations (direction as well as magnitude) with respect to position in space. The term $\partial u_i / \partial t$ is called "local" acceleration and refers to the variation of u_i (at a particular point in space) with respect to time.

3.1.3. Fluid Shear

A resisting force acting tangentially to the surface of moving particles is called shear. It is proportional to the area over which it acts, and depends on the component of velocity gradient normal to the plane in which it acts. Shear on a particular face of a fluid element (say an element consisting of a cube) is a force in the direction of motion of a faster moving adjacent element. If the motion of the adjacent element is slower, the force of shear is in the opposite direction on the face under consideration. The force/area is called intensity of shear τ.

Shear is a tensor quantity. In order to specify τ at a particular point, it is necessary to identify the orientation of the plane under consideration and the direction within this plane as well as its magnitude. Thus, τ_{ij} means the intensity of shear (at a particular point) in a plane normal to i in the direction j.

In analyzing the resultant force on a volume element, say a cube, it is necessary to consider only the shear at the faces of the cube. This is because shear on other planes within the cube exerts equal force in opposite directions on adjacent parts of the same element; that is, the resultant effect of shear on all but the surface faces is zero in respect to a particular element under consideration.

The resultant of shear on a fluid particle of any shape can be expressed generally by

$$\mathbf{F} = \int_{A_p} \boldsymbol{\sigma} \times \mathbf{dA} \qquad 3.3$$

in which $\boldsymbol{\sigma}$ is the surface force, $\mathbf{dA}$ is a vector having the direction of the outer normal to the surface and the magnitude of a differential segment of the surface, and A_p is the entire surface area of the particle. Note that $\boldsymbol{\sigma}$ is not τ, because it may have a normal as well as a tangential component. The cross product $\boldsymbol{\sigma} \times \mathbf{dA}$, however, evaluates only the tangential components. Unlike σ_c appearing in Equation 2.1, $\boldsymbol{\sigma}$ is not conservative, and cannot be related to volume and temperature in an equation of state. Furthermore **F**, unlike pressure as defined by Equation 2.1, is a vector.

Although, Equation 3.3 is useful for visualizing the effect of shear in creating a resultant force on a volume element, it does not help in evaluating this force. A more useful expression for F_i, the component of shear in the i direction, can be developed in respect to the spatial distribution of velocity. A derivation of the latter expression, when a velocity component is varying in three coordinate directions, is beyond the scope of this text. Such a derivation is given in all advanced texts on fluid mechanics. However, a derivation of F_i, for the simple case of 1-dimensional flow, provides a relationship in the same form as the general relationship and may provide some insight into the way shear acts.

A case is considered in which u_i is varying in an orthogonal direction j and components u_j and u_k are zero. The space coordinate in the j direction is x_j. In this case it is assumed that

$$\tau_{ji} = \mu \frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j},$$

in which μ is the viscosity of the fluid. The partial derivative $\partial u_i / \partial x_j$ represents the rate of angular deformation (in planes normal to k) of volume elements of the fluid. The assumption that the partial derivative is linearly related to the shear stress means that the fluid is a Newtonian viscous fluid. For such a fluid, in the general case of 3-dimensional flow, it is possible to show that

$$\tau_{ji} = \mu \left(\frac{\partial \mathbf{u}_j}{\partial \mathbf{x}_i} + \frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j} \right) .$$

However, in the case under consideration, the first term in the parenthesis is zero because u_j is zero.

A particle having the shape of a cube with a volume $(\delta\ell)^3$ is considered, as illustrated in Figure 3.1.

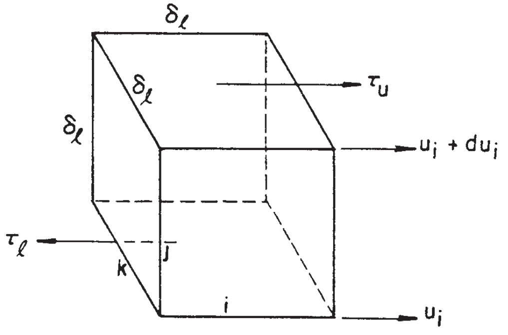

Figure 3.1. Shear on a cube undergoing 1-dimensional deformation.

The direction of shear in the upper face τ_u is indicated in Figure 3.1, as is the direction of shear in the lower face τ_ℓ. These directions imply that du_i, also shown in the figure, is positive. In other words, $\partial u_i / \partial x_j$ is positive for the case illustrated.

For this case,

$$\mathbf{F_i} = \left[\left(\tau_{\mathbf{ji}}\right)_{\mathbf{u}} - \left(\tau_{\mathbf{ji}}\right)_{\ell}\right] \frac{(\delta\ell)^2}{(\delta\ell)^3} \cdot$$

Since

$$\tau_{\mathbf{ji}} = \mu \frac{\partial \mathbf{u_i}}{\partial \mathbf{x_j}},$$

the term in brackets becomes

$$\mu \left[\frac{\partial}{\partial \mathbf{x_j}} \left(\frac{\partial \mathbf{u_i}}{\partial \mathbf{x_j}}\right)\right] \delta\ell$$

and

$$\mathbf{F_i} = \mu \frac{\partial^2 \mathbf{u_i}}{\partial \mathbf{x_j}\, \partial \mathbf{x_j}} \cdot \tag{3.4}$$

Although Equation 3.4 was derived for a case in which all velocity components except u_i are zero, and u_i is varying only in the j direction, it has the same form as the expression for the case of 3-dimensional flow. In the latter case

$$\frac{\partial^2 \mathbf{u_i}}{\partial \mathbf{x_j}\, \partial \mathbf{x_j}} \equiv \frac{\partial^2 \mathbf{u_i}}{\partial \mathbf{x_i}\, \partial \mathbf{x_i}} + \frac{\partial^2 \mathbf{u_i}}{\partial \mathbf{x_j}\, \partial \mathbf{x_j}} + \frac{\partial^2 \mathbf{u_i}}{\partial \mathbf{x_k}\, \partial \mathbf{x_k}}$$

which conforms with the summation convention.

In vector notation,

$$\frac{\partial^2 \mathbf{u_i}}{\partial \mathbf{x_j}\, \partial \mathbf{x_j}} \equiv \nabla^2 \mathbf{u_i}\,.$$

3.1.4. Equation of Fluid Motion

Substituting Equation 3.2 and 3.4 into Equation 3.1 gives

$$\rho \left[u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial t} \right] = \rho g_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \, \partial x_j}. \qquad 3.5$$

Equation 3.5 is the Navier-Stokes equation of fluid motion for Newtonian viscous fluids undergoing negligible divergence.

Equation 3.5 can be simplified further for application to flow in porous media. If the medium is homogeneous, the convective acceleration

$$u_j \frac{\partial u_i}{\partial x_j},$$

when integrated over a macroscopic volume, is zero for uniform, rectilinear macroscopic flow [Hubbert (1940)]. The velocity (statistically) is unchanged in respect to both magnitude and direction as a result of fluid passing through a macroscopic volume element of the medium. If u_i is small, $\partial u / \partial t$ is also small. Usually both local and convective terms are small even for non-homogeneous porous media or for flow that is not uniform or rectilinear. Neglecting the terms in Equation 3.5 that are insignificant for flow in porous media gives

$$\frac{\partial p}{\partial x_i} - \rho g_i = \mu \frac{\partial^2 u_i}{\partial x_j \, \partial x_j}. \qquad 3.6$$

Both Equation 3.5 and 3.6 are written in respect to fluid particles totally within a single fluid phase. Elements that include some of each phase are excluded.

3.2. POTENTIALS

It is sometimes convenient to express force components acting on fluid particles as negative gradients of scalar quantities called force potentials. Potentials are related to potential energy, implying a capacity to do work. However, it is possible to define force potentials only in respect to force fields that are conservative [Hubbert (1940)].

The potential energy of an element of fluid (with respect to a particular field of force) is a function of its position in the field. The value of the potential (when it exists) is a measure of the work that would be done by a particular force field on a fluid particle in moving from a given position to a datum position, providing this work could be done reversibly.

The existence of a potential of a particular force field does not depend upon whether or not a fluid particle physically could be moved from one point to another reversibly. The existence depends, instead, upon the way the force is related to the space coordinates. A force field for which a scalar potential can be defined is said to be conservative.

A necessary and sufficient condition for a force field to be conservative is that the work integral

$$\oint \mathbf{f} \bullet \mathbf{ds} = 0$$

where **f** is the force considered, and **ds** is a differential displacement vector in the field. The circle indicates that the integral has been taken about any closed path. If this condition is satisfied, it is possible to define a scalar Φ by

$$\Phi \equiv \int_{s_o}^{s} -\mathbf{f} \bullet \mathbf{ds} \qquad 3.7$$

where Φ is the potential at the position **s**, and $\mathbf{s_o}$ represents the position of a datum in respect to which Φ is evaluated. The negative sign of the integrand conforms with the standard sign convention for potentials. The definition is ambiguous if the work integral around every closed path is not zero.

3.2.1. Pressure Potential

The significance of a potential (when its definition is valid) is that the negative gradient of the potential represents the force acting at any point in the system. Forces that are a consequence of motion (in this case shear) and result in the dissipation of energy in the form of heat are non-conservative. Driving forces (those that tend to produce the motion) are sometimes conservative. For example,

$$p_s \equiv \int_{s_o}^{s} - \left[-\frac{\partial p}{\partial s} \right] ds,$$

if p_{so} is set equal to zero. The value of the line integral is independent of the path so the definition of p_s is not ambiguous. In this case the force field is conservative and a potential exists.

Pressure, therefore, is a potential having the dimensions of energy/volume, the negative gradient of which is the force per unit volume resulting from the spatial distribution of pressure. Whereas p is a scalar, the gradient of p,

$$\frac{\partial p}{\partial x_i} e_i \, ,$$

is a vector and as such has a particular direction associated with it, depending upon the relative magnitude of its components,

$$\frac{\partial p}{\partial x_i} .$$

In physics, it is customary to define force potentials with the dimensions of energy/mass. However, a pressure potential having dimensions of energy/mass does not exist for the general case. If such a potential existed it would be defined by

$$\Phi_p \equiv \int_{s_o}^{s} - \left[- \frac{\partial (p/\rho)}{\partial s} \right] ds .$$

The integral is path independent only if ρ is constant with respect to **s**, or varies only with p. Cases where density varies only with pressure are called "barotropic" cases.

The error resulting from regarding p/ρ as a force potential (for a case of ρ varying independently with the space coordinates) can be understood by arbitrarily defining Φ as p/ρ. Note that

$$-\frac{\partial (p/\rho)}{\partial x_i} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} - p \frac{\partial (1/\rho)}{\partial x_i} .$$

Clearly, the component of force in the i direction (due to the pressure gradient) is $-(1/\rho)(\partial p / \partial x_i)$, but the second term on the right is not a force component. It is an error term resulting from arbitrarily defining Φ as p/ρ.

M.K. Hubbert (1940) interpreted the pressure potential as representing potential energy of compression, that is, work done against compressive forces in increasing the pressure of a fluid element from p_0 to p. In fact, pressure potentials have no relation to compressibility. Evidence for this is the fact that the coefficient of compressibility does not appear in the integral defining Φ_g. A correct interpretation of Φ_g is the work done in moving a reference element of fluid from a datum position to a position under consideration against forces generated by pressure gradients. Unfortunately, the misinterpretation of pressure potentials, apparently first introduced by Hubbert (1940), has been repeated in many textbooks dealing with fluid flow, especially in textbooks dealing with groundwater hydraulics.

3.2.2. Gravitational Potential

The potential Φ_g due to gravity is given by

$$\mathbf{\Phi_g \equiv \int_{s_0}^{s} -g \bullet ds}$$

where **g** is the force/mass due to gravity. Where the elevation difference is small enough that changes in **g** can be ignored,

$$\mathbf{\Phi_g = gh,}$$

where h is the elevation difference between $\mathbf{s_o}$ and **s**. Here Φ_g has the dimensions of energy/mass.

A gravity potential having the dimensions of energy/volume is defined by

$$\mathbf{\Phi_{gv} \equiv \rho gh} \; ,$$

in which g is a scalar representing the magnitude of **g**. Taking the negative derivative of $\mathbf{\Phi_{gv}}$ with respect to x_i gives

$$\mathbf{-\frac{\partial \Phi_{gv}}{\partial x_i} = -\rho g \frac{\partial h}{\partial x_i} - gh \frac{\partial \rho}{\partial x_i}} \; ,$$

providing g is a constant. The first term on the right, is a valid expression for the component of force/volume due to gravity, but the second term is an error term having no relation to a force component.

The magnitude of $-gh\left(\partial\rho/\partial x_i\right)$ depends on the elevation h as well as on the component of the density gradient. Its importance depends on its magnitude relative to that of $-\rho g\left(\partial h/\partial x_i\right)$ and $-\left(\partial p/\partial x_i\right)$. A potential evaluated as ρgh, therefore, should be used with caution, and in particular, it should not be assumed that the error is unimportant when the variation in density is small. The error, if the system is large, can be important even with small density differences, such as, when $\partial h/\partial x_i$ and $\partial p/\partial x_i$ are both of the same order as $\partial\rho/\partial x_i$.

3.2.3. Combined Potentials

When it is possible to define two or more force potentials in respect to their individual force fields, it is permissible to add the potentials to obtain a total potential Φ_t. The negative gradient of Φ_t gives the combined force component in the direction i. It is impossible to combine Φ_p and Φ_g, regardless of the dimensions in which these potentials are expressed, unless ρ is constant or depends only on p. Either Φ_p or Φ_g is undefined in the general case. For the special case of a constant density, however, it is very convenient to combine Φ_p and Φ_g. This requires that both component potentials have the same dimensions and refer to the same reference element. Two ways of doing this are in common use:

(1) A potential p^* is defined by

$$\mathbf{p^* \equiv p + \rho g h} \qquad 3.8$$

where the dimensions are energy/volume. The potential p^* is called piezometric pressure.

(2) A potential H is defined by

$$\mathbf{H \equiv \frac{p}{\rho g} + h} \qquad 3.9$$

where the dimensions are energy/weight or length. H is called piezometric head.

Occasionally some authors have used a third combined potential,

$$\mathbf{\Phi_M \equiv \frac{p}{\rho} + gh\ .} \qquad 3.10$$

having the dimensions of energy/mass, but the latter is seldom used in either hydraulics or fluid mechanics.

Combined or "total potentials" should be used with caution. Many early investigators, for example, Richards (1928), called piezometric pressure a "total" potential. This is a valid concept provided ρ is constant. In later years, however, some investigators [notably Edlefsen and Anderson (1943)] attempted to extend the concept of total potential to include terms not pertaining to forces on fluid particles as defined in this text. For example, they included terms pertaining to the motion (of individual species of molecules, e.g., water) in response to osmotic and temperature effects. As pointed out by Corey and Kemper (1961), this procedure led to erroneous conclusions that have been widely disseminated among soil scientists and plant physiologists. A "total" potential is defined only if:

(1) All force fields for which potentials are defined are conservative.

(2) All component potentials are in respect to a common reference element. It is not valid, for example, to add an osmotic potential (in respect to water molecules) to a potential relating to fluid particles.

(3) All potentials are force potentials in the Newtonian sense. For example, so-called "forces" pertaining to molecular diffusion in response to temperature and concentration gradients have no independent effect on the acceleration of the center of mass of a fluid particle.

Convective and diffusion processes should be treated as separate transport mechanisms, and the fluxes combined after each mechanism has been evaluated independently [A. T. Corey and A. Klute (1985)].

3.3. VISCOUS FLOW

Equation 3.6 is a simplified form of the Navier-Stokes equation applicable where divergence of the velocity vector and fluid inertia are negligible. Flow under such circumstances is called "viscous flow," implying that the resistance force is due only to fluid viscosity. Writing Equation 3.6 in terms of the potential function p^* results in

$$\frac{\partial p^*}{\partial x_i} = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \qquad 3.11$$

which is a valid approximation for flow in porous media under ordinary potential gradients provided the fluid density is constant.

Equation 3.11 is applicable to flow through the pore space of porous media. However, it is necessary to describe the geometric boundaries of the pore space mathematically to solve the differential equation. The pore space, however, has a very complex geometry. Even if the complex geometry could be described, it is doubtful that the equation could be solved for such boundaries. The approach used in the following analyses is to select highly idealized and simplified models involving viscous flow, and to solve Equation 3.11 for these cases. By this procedure, an insight is provided into the behavior of fluids in porous media.

In addition to the assumptions, previously discussed, that were accepted for the derivation of Equation 3.11, additional assumptions are made in describing the boundary conditions for the flow. These are:

(1) At boundaries where a fluid is in contact with a solid, fluid velocity relative to the boundary is zero. If this were not true, derivatives of u with respect to the normal to boundaries would be infinite and an infinite shear at boundaries would be implied. This is not in conformance with the continuum assumption that velocity is an analytic function of the space coordinates at all points including the boundaries.

(2) At boundaries where a liquid is in contact with a gas, shear is assumed to be negligible. This assumption is based on the fact that the viscosity of common liquids are more than 50 times the viscosity of common gases.

(3) Symmetry of velocity distribution is assumed where there is no reason to postulate a lack of symmetry.

3.3.1. Flow in a Film

The first model considered is a liquid flowing steadily over a flat solid boundary in a film of uniform thickness d. The flow is 1-dimensional in the direction i; that is, u_j and u_k are zero. Such a model is illustrated in Figure 3.2, in which a section of the film (in a plane parallel to i and normal to the solid boundary) is shown.

Because all components of **u** orthogonal to u_i are zero, the right side of Equation 3.11 can be written as an ordinary second derivative, that is

$$\mu \frac{d}{dy}\left(\frac{du}{dy}\right),$$

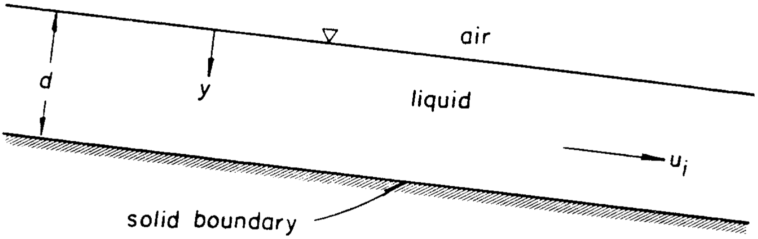

Figure 3.2. Flow in a film over a flat solid boundary.

in which y is a coordinate measured from the surface of the film in a direction normal to the solid boundary and also to the direction of u. The subscript i is dropped because **u** has zero components in orthogonal directions. Likewise, the left side of Equation 3.11 can be written as an ordinary derivative since there is no flow and no component of the potential gradient orthogonal to i. Letting x be the coordinate in the i direction results in

$$\frac{dp^*}{dx} = \mu \frac{d}{dy}\left(\frac{du}{dy}\right) .$$

The piezometric pressure does not vary with y because there is no flow normal to the boundary. Consequently, the derivative of piezometric pressure with respect to x can be treated as a constant when integrating with respect to y. Integrating with respect to y gives

$$\frac{dp^*}{dx} y = \mu \frac{du}{dy} + c .$$

Since τ at an air boundary is assumed to be zero

$$\frac{du}{dy} = 0 \quad \text{at} \quad y = 0 ,$$

therefore

$$\frac{y}{\mu} \frac{dp^*}{dx} = \frac{du}{dy} .$$

Integrating a second time gives

$$\frac{y^2}{2\mu}\frac{dp^*}{dx} = u + c' .$$

Evaluating c' on the basis that u is zero at the solid boundary gives

$$u = \frac{-1}{2\mu}\frac{dp^*}{dx}\left(d^2 - y^2\right) \qquad 3.12$$

which indicates that the velocity distribution is parabolic.

The average velocity $\bar{u}$ is defined by the expression

$$\bar{u} \equiv \frac{1}{d}\int_o^d u\,dy .$$

Substituting Equation 3.12 into this expression and integrating gives

$$\bar{u} = -\frac{d^2}{3\mu}\frac{dp^*}{dx} . \qquad 3.13$$

The discharge through the film per unit width normal to the flow is given by $\bar{u}d$.

3.3.2. Flow Through Slits

The second case considered also involves 1-dimensional flow. The model is like that for the film except that the upper flow boundary, as well as the lower, is solid. This model is illustrated in Figure 3.3. Again a

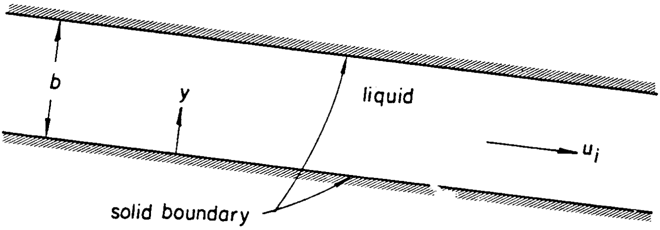

Figure 3.3. Flow through a slit of uniform thickness.

section normal to the boundaries and parallel to the flow is considered. The coordinate in the direction of flow is designated as x, and the coordinate orthogonal to both x and the boundaries is y. The origin for y is at one of the solid boundaries.

The flow equation reduces to an ordinary differential equation as for the case of flow over a plate. Integrating with respect to y twice, and using the assumption, that because of symmetry, du/dy is zero where y is b/2, and that u_i is zero where y is either zero or b, the result is

$$u = \frac{1}{2\mu}\left(y^2 - by\right)\frac{dp^*}{dx} . \qquad 3.14$$

Also,

$$\bar{u} = -\frac{b^2}{12\mu}\frac{dp^*}{dx} . \qquad 3.15$$

3.3.3. Flow Through Tubes with Circular Cross-sections

In the case of flow through circular tubes, Equation 3.11 cannot be reduced to an ordinary differential equation in an orthogonal coordinate system. However, in cylindrical coordinates it is an ordinary equation, assuming the flow is symmetrical about the axis of the tube. Such a tube is shown in Figure 3.4.

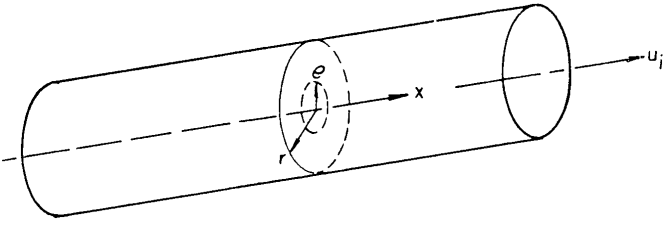

Figure 3.4. Flow through a capillary tube.

In cylindrical coordinates, the operator

$$\frac{\partial^2}{\partial x_j \partial x_j}$$

becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial x^2},$$

in which ρ is the radial coordinate with origin at the axis. For the symmetrical case, the two terms on the right are zero. Consequently,

$$\frac{dp^*}{dx} = \mu\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)\right].$$

Again using the argument that $du/d\rho$ is zero at the center and u is zero at the solid boundary, the result is

$$\bar{u} = -\frac{r^2}{8\mu}\frac{dp^*}{dx}, \qquad 3.16$$

where r is the radius of the tube. Equation 3.16 is known as Poiseuille's equation.

3.3.4. Generalized Equation for Flow Through Straight Conduits

By induction, after examining Equations 3.13, 3.15 and 3.16, one may conclude that an equation of the form

$$\bar{u} = -\frac{D^2}{k_s\,\mu}\frac{dp^*}{dx} \qquad 3.17$$

describes the average flux rate for small uniform conduits in general. In Equation 3.17, D is a characteristic length dimension describing the size of cross-section and k_s is a factor depending on the shape and the dimension selected. For example k_s is 32 instead of 8 if the diameter had been used instead of the radius to describe the size of tube shown in Figure 3.4.

Equation 3.17 states that the average velocity (and, consequently, the discharge) is proportional to the driving force in a small straight tube. It also indicates that the constant of proportionality is directly related to a characteristic length squared and is inversely related to fluid viscosity.

3.4. HYDRAULIC RADIUS

Since the factor k_s in Equation 3.17 depends on the choice of a length dimension to describe the tube size, it is desirable to select a dimension that can be defined in an unambiguous way for all cross-sectional shapes. It is also desirable to select a dimension that correlates well with discharge for a given driving force and minimizes the range of k_s.

The length used to describe the tube size is the "hydraulic radius" R defined by

$$\mathbf{R \equiv \frac{A}{wp}} \tag{3.18}$$

in which A is the area of cross-section normal to flow and wp is the length of the wetted perimeter of the flow section in the plane of A.

For the tubes analyzed in Sections 3.3.1, 3.3.2 and 3.3.3, the value of R is given by d, b/2 and r/2 respectively. By replacing the dimensions d, b and r with their equivalent in terms of R, the equations for average velocity for the film and slit become

$$\bar{u} = -\frac{R^2}{3\mu}\frac{dp^*}{dx} , \tag{3.19}$$

and that for the circular tube becomes

$$\bar{u} = -\frac{R^2}{2\mu}\frac{dp^*}{dx} . \tag{3.20}$$

For films, slits, and tubes, use of the hydraulic radius as a characteristic length reduces the range of k_s to a factor varying from 2 to 3.

However, the use of R does not reduce the range of the shape factor for all shapes. For example, consider a tube of about the same cross-sectional area as that shown in Figure 3.4, but with a shape as illustrated in Figure 3.5.

For a case of laminar flow of a viscous fluid, $\bar{u}$ is not greatly different from $\bar{u}$ for a tube with a smooth bore of the same cross-sectional area. As long as turbulence is not a factor, the spatial distribution of velocity in the main part of the tube is affected only slightly by the local variations near the perimeter. Yet the value of R may be smaller by a large factor in the tube with the extended perimeter. It is clear that the hydraulic radius must be used with caution.

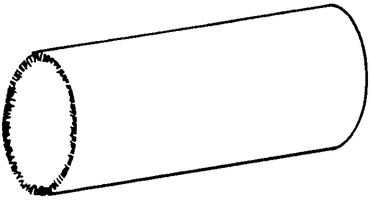

Figure 3.5. Cross-section of tube with extended wetted perimeter.

3.4.1. Effect of Tube-size Distribution

One way to overcome the problem posed by shapes analogous to that shown in Figure 3.5 is to regard the flow region in the vicinity of the rough perimeter as being separate from the main portion of the tube. It is theoretically possible to evaluate an R for each little crevice around the boundary, and to sum the flow contributions of the crevices, which in the case illustrated would be negligible. An analogous procedure is used in hydraulics in evaluating flow through river channels with irregular cross-sections.

In a similar way, it is possible to evaluate the flow through a model consisting of straight circular tubes of varying diameter as shown in Figure 3.6. Assuming that Equation 3.20 applies to each individual tube, it is possible to find a valid value for $\bar{u}$ for the model as a whole by adding the contribution to the discharge of each individual tube and then dividing by

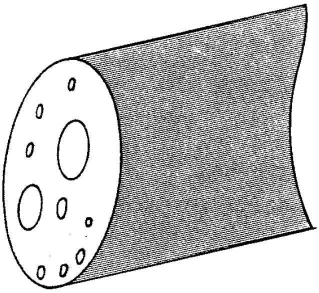

Figure 3.6. Cylinder of solid material with a number of round holes bored parallel to its long axis.

the total cross-sectional area. Applying Equation 3.20 to flow through each hole in this model results in

$$Q_i = -\frac{4\pi R_i^4}{2\mu}\frac{dp^*}{dx} ,$$

where Q_i is the discharge through a particular hole with hydraulic radius R_i.

Consequently, for the bundle as a whole,

$$Q = \sum_{i=1}^{n} Q_i = -\frac{4\pi}{2\mu}\frac{dp^*}{dx}\sum_{i=1}^{n} R_i^4 .$$

Using the relationship that the total cross-sectional area of the bulk system is equal to the total pore area divided by ϕ gives

$$q = -\frac{\phi\overline{R^2}}{2\mu}\frac{dp^*}{dx} \qquad 3.21$$

in which q is the ratio Q/A, A is the total cross-sectional area of the cylinder, and

$$\overline{R^2} \equiv \frac{\sum_{i=1}^{n} R_i^4}{\sum_{i=1}^{n} R_i^2}$$

The quantity $\overline{R^2}$ is a weighted mean value of R^2.

In general, $\overline{R^2}$ is larger than the square of a mean value of R given by the ratio of total tube volume to the total internal surface area of a tube. This can be verified by considering a model with two holes with R_i equal to 1 and 3 respectively. For this case, $(\overline{R})^2$ is 6.25 whereas $\overline{R^2}$ is 8.2. The disparity becomes much greater as the ratio of larger to smaller R_i values increases. The disparity is enormous when the R_i differ by orders of magnitude.

3.4.2. Hydraulic Radius Related to Pore size

As explained in Section 3.4.1, an expression which gives an average value of R, for a tube of non-uniform cross-section, is the internal volume of the tube divided by its internal surface area. A statistically equivalent expression applicable to a porous medium is ϕ / s. If the volume of the reference element is as small as a portion of the pore space between individual grains, the volume divided by the internal surface approaches the value of R for that portion of the pore space. As explained in Section 2.4, one way to evaluate A/wp for a portion of pore space is to assume that it is inversely related to the value of capillary pressure when an interface is stretched across the section of pore space under consideration.

Capillary pressure, therefore, should be a measure of the largest pore-size or R value for that part of the pore space occupied by the wetting fluid at a particular saturation. It is also a measure of the smallest R value of that part of the pore space occupied by the nonwetting fluid.

3.4.3. Value of $\overline{R^2}$ as a Function of S

As S decreases, the maximum R for the wetting phase (or minimum R for the nonwetting phase) also decreases. The size of the pore space is characterized by R for all of the space occupied by a particular fluid. For the wetting phase,

$$\overline{R_w^2} = \frac{\sigma^2 \cos^2 \alpha}{p_d^2 S} \int_0^S \left(\frac{p_d}{p_c} \right)^2 dS \, , \qquad 3.22$$

and for the nonwetting phase,

$$\overline{R_{nw}^2} = \frac{\sigma^2 \cos^2 \alpha}{p_d^2 (1 - S)} \int_S^1 \left(\frac{p_d}{p_c} \right)^2 dS \, . \qquad 3.23$$

Equations 3.22 and 3.23 can be modified for a 3-phase fluid system. Equations similar to Equations 3.22 and 3.23 were first proposed by Purcell (1949). Purcell based his derivation on a model of a porous medium consisting of a bundle of capillary tubes.

3.5. TORTUOSITY

A porous medium is not a bundle of straight tubes, or even a bundle of sinuous tubes. An appropriate model must consist of a network of interconnected channels. Fluid particles flowing through granular porous media follow a sinuous path. Although in granular media the pore space is interconnected, some aspects of the effect of the sinuous path can be deduced by considering a model consisting of a single sinuous channel, as shown in Figure 3.7.

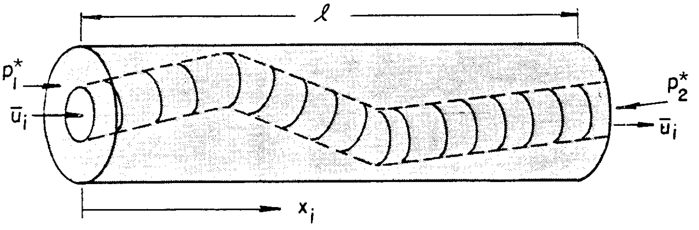

Figure 3.7. Solid with a single sinuous channel.

It is possible to measure the length ℓ in the direction i from the point where the piezometric pressure is p_1^* to a point where it is p_2^*. However, it is not possible to determine directly the distance ℓ that fluid particles must move in passing from point 1 to point 2. The path length is designated by ℓ_e, implying this is the effective distance of flow between point 1 and point 2. The fact that ℓ_e / ℓ is greater than 1.0 has two important consequences:

(1) The average value of $-\partial p^* / \partial \ell_e$ acting on the fluid is

$$\frac{p_1^* - p_2^*}{\ell_e}$$

or

$$\frac{\ell}{\ell_e}\left(\frac{p_1^* - p_2^*}{\ell}\right) .$$

(2) The average component of velocity in the i direction is

$$\bar{u}_i = \frac{\ell}{\ell_e}\bar{u}_t ,$$

in which $\bar{u}_t$ is the average velocity tangential to the flow path.

Calculating $\bar{u}_t$ (knowing only R for the tube and the values of p_1 and p_2), gives

$$\bar{u}_t = \frac{R^2}{k_s\mu}\left(\frac{p_1^* - p_2^*}{\ell_e}\right)$$

or

$$\bar{u}_i \left(\frac{\ell_e}{\ell}\right) = \frac{R^2}{k_s \mu} \frac{\ell}{\ell_e} \left(\frac{p_1^* - p_2^*}{\ell}\right) .$$

In differential form this is written as

$$\bar{u}_i = -\frac{R^2}{k_s T \mu} \frac{\partial p^*}{\partial x_i} , \qquad 3.24$$

in which T is called tortuosity $(\ell_e / \ell)^2$. The ratio ℓ / ℓ_e may also be interpreted as the average cosine of the angle of $\bar{u}_t$ with respect to the direction i.

According to Wyllie and Spangler (1952), T for typical granular porous media, when fully saturated with a single fluid, is about 2, which implies that the average angle of the fluid path to the macroscopic flow direction is about 45 degrees. This also was observed to be a good approximation by Sullivan and Hertel (1942). By assuming a random-walk model with the preferred angle being zero, and the probability decreasing to zero at an angle of π, it is possible to deduce that the average angle is $\pi/4$ which agrees with the observation that the value of T is about 2 for the fully saturated case. Values of T may differ from 2 significantly, however, for non-isotropic media.

3.5.1. Tortuosity as a Function of Saturation

As the wetting-phase saturation of a porous medium decreases, fluid particles of the wetting phase must take an increasingly longer path in moving between two points. This is because the particles cannot take a direct route across pore spaces because the central portion of the spaces are occupied by the nonwetting phase. After sufficient desaturation takes place, the nonwetting phase has a finite tortuosity, and this tortuosity decreases with decreasing liquid saturation.

Burdine (1952) and Corey (1954) found that the tortuosity of a wetting phase T is inversely related to S_e^2; that is,

$$\left(\frac{T_{1.0}}{T_S}\right)_w \approx S_e^2 \qquad 3.25$$

in which $T_{1.0}$ is the tortuosity of the wetting phase when S_e is 1.0. The same investigators also found that for the nonwetting phase

$$\left(\frac{T_{1.0}}{T_S}\right)_{nw} \approx (1 - S_e)^2 \; . \qquad 3.26$$

Both Equation 3.25 and 3.26, however, are valid only for isotropic media. Wyllie and Gardner (1958) later deduced the same $T(S_e)$ functions from probability considerations using a capillaric model for a porous medium consisting of randomly interconnected bundles of tubes.

3.6. KOZENY-CARMAN EQUATION

Kozeny (1927), and later Carman (1937) and Fair and Hatch (1933) developed an equation for $\bar{u}_i$ for a fully-saturated granular media by substituting the average value of R (given by ϕ/s) into Equation 3.24. Noting that the discharge Q through an area A of the porous medium normal to i is $\bar{u}_i\phi$, they obtained

$$q_i = -\frac{k}{\mu}\frac{\partial p^*}{\partial x_i}$$

in which

$$k \equiv \frac{\phi^3}{k_s T s^2} \; , \qquad 3.27$$

and q_i is Q/A. Equation 3.27 is known as the Kozeny-Carman equation for permeability. Kozeny entered only one constant in his original equation and determined that it was about 5. Carman pointed out that Kozeny's constant was actually a product of two factors, that is, k_s and T. However, Carman agreed that the product was about 5, since he thought k_s to be about 2.5 and T to be about 2.0 for sand beds. The equation, with these constants, has been used in some industries as an indirect method of determining specific surface for granular powders and casting sands.

The same constants have been found for sands by many later investigators, but the relationship fails completely when applied to something other than sands. This is not surprising in view of the discrepancy between $(\overline{R})^2$ and $\overline{R^2}$ which was discussed in Section 3.4.1. The Kozeny-Carman equation also cannot be applied to media less than fully saturated with a single fluid.

3.6.1. Generalized Kozeny-Carman Equation

A version of Equation 3.24 should be valid for porous media in general, provided account is taken of the range of values of R due to the distribution of pore-sizes. A relationship valid for S < 1 can be obtained by accounting also for the changes in $\overline{R^2}$ and T with S.

An examination of Equation 3.22 indicates that a more valid expression for $\overline{R_w^2}$ might be obtained by averaging R_w^2 over the range of saturations $(S > S_r)$ that actually contributes significantly to flow. This gives

$$\overline{R_w^2} = \frac{\sigma^2 \cos^2 \alpha}{p_d^2 S_e} \int_o^{S_e} \left(\frac{p_d}{p_c} \right)^2 dS_e \, .$$

The value of the integral is not changed significantly by the substitution of S_e for S because the ratio p_d/p_c approaches zero for $S < S_r$. However, $\overline{R_w^2}$ is made slightly larger because $1/S_e$ is slightly larger than 1/S. This is because R_w^2 is averaged over only that part of the pore space in which R has a size large enough to permit significant flow. Observing that q_i for the wetting phase is $\bar{u}_i \phi_e S_e$, and employing the expression for $\overline{R_w^2}$ and Equation 3.25 for $T_w(S_e)$ results in

$$q_{wi} = -\frac{k_w(S_e)}{\mu_w} \frac{\partial p^*}{\partial x_i} \qquad 3.28$$

in which

$$k_w(S_e) = \frac{\phi_e \sigma^2 \cos^2 \alpha}{p_d^2 k_s T_{1.0}} S_e^2 \int_o^{S_e} \left(\frac{p_d}{p_c} \right)^2 dS_e \, .$$

Similarly,

$$q_{nwi} = -\frac{k_{nw}(S_e)}{\mu_{nw}} \frac{\partial p^*}{\partial x_i} \, , \qquad 3.29$$

in which

$$k_{nw}(S_e) = \frac{\phi_e \sigma^2 \cos^2 \alpha}{p_d^2 k_s T_{1.0}} (1 - S_e)^2 \int_{S_e}^{1} \left(\frac{p_d}{p_c} \right)^2 dS_e \, .$$

Equations 3.28 and 3.29 are analogous to equations first proposed by Burdine (1952). They may be regarded as generalized Kozeny-Carman equations, a name proposed by Wyllie and Spangler (1952) for related equations. They are not valid for non-isotropic media, because the analysis assumes that the pore space has no preferred orientation.

Equations 3.28 and 3.29 were greatly simplified by Brooks and Corey (1964) by substituting Equation 2.9 into the integrals and performing the indicated integration. They obtained

$$\mathbf{k_w = k\,,} \quad \text{for} \quad \mathbf{p_c \le p_d}$$

and

$$\mathbf{k_w = k\left(\frac{p_d}{p_c}\right)^{\eta}\,,} \quad \text{for} \quad \mathbf{p_c \ge p_d} \qquad 3.30$$

in which k is the value of k_w when S_e is 1.0, and

$$\boldsymbol{\eta = 2 + 3\lambda\,.} \qquad 3.31$$

In terms of S, they obtained

$$\mathbf{k_w = k S_e^{\varepsilon}} \qquad 3.32$$

in which

$$\boldsymbol{\varepsilon = \frac{2 + 3\lambda}{\lambda}\,.} \qquad 3.33$$

For the nonwetting phase, they obtained

$$\mathbf{k_{nw} = k(1 - S_e)^2\,(1 - S_e^{\gamma})} \qquad 3.34$$

for $S < S_m$, in which S_m is some maximum value of S_e at which k_{nw} exists, and γ is $(2 + \lambda)/\lambda$. The value of S_m is usually about 0.85 for homogeneous and isotropic materials.

Equations 3.30 through 3.34 were verified experimentally by Brooks and Corey (1966) and by Laliberte et al. (1966). Laliberte et al. (1968) obtained an expression for k by evaluating the integrals in Equation 3.28 between the limits of S_e from 0 to l, to obtain

$$\mathbf{k} = \frac{\phi_e \sigma^2 \cos^2 \alpha}{k_s T_{1.0}\, p_d^2\, \gamma} \qquad 3.35$$

in which γ is the same parameter appearing in Equation 3.34. Since Laliberte et al. used a light hydrocarbon liquid with a negligible angle of contact in obtaining $p_c(S)$, they set cos α equal to 1. They also accepted the published approximations for k_s and $T_{1.0}$, so that

$$\mathbf{k} \approx \frac{\phi_e \sigma^2}{5\gamma\, p_d^2} \; . \qquad 3.36$$

Combining Equation 3.36 with Equation 3.32 gives

$$k_w\left(S_e\right) \approx \frac{\phi_e \sigma^2}{5\gamma\, p_d^2}\, S_e^{\varepsilon} \; . \qquad 3.37$$

3.7. PERMEABILITY

The analysis beginning in Section 3.1 by considering the motion of fluid particles leads finally to Equation 3.28, a macroscopic flux equation. That is, q_{wi} is a volume flux rate averaged over an element of area that includes solid as well as pore space. The fluid does not actually flow through all of the area, and q_{wi} does not represent the seepage velocity $\bar{u}_i$. The relationship between q_{wi} and $\bar{u}_i$ is given by

$$q_{wi} \approx \phi_e\, S_e\, \bar{u}_i \; . \qquad 3.38$$

The relationship between q_{wi} and the average velocity of particles tangential to their flow path is approximated by

$$q_{wi} \approx \phi_e\, S_e \frac{\ell}{\ell_e}\, \bar{u}_t \approx \phi_e \frac{S_e^2}{\sqrt{2}}\, \bar{u}_t \; . \qquad 3.39$$

Consequently, q_{wi} is not a velocity component of the center of mass of a fluid particle. Equation 3.28 is more accurately called a flux equation rather than a flow equation.

The proportionality function $k_w(S_e)$ in Equation 3.28 is called effective permeability in the petroleum literature. It is a function of the geometric properties of the pore space occupied by the wetting fluid under consideration. It has the dimensions of L^2. In the petroleum literature, permeability (when used without adjectives) refers to the value of k_w when S is 1.0. This is often designated simply by k as was done in Equations 3.30 and 3.34. The ratio k_w/k is called relative permeability to the wetting phase and is designated by k_{rw}. Likewise k_{nw}/k is designated by k_{rnw}.

3.7.1. Factors Affecting Permeability

According to Equations 3.32 and 3.34, the maximum value of both k_w and k_{nw} is k. There are factors which are not taken into account in the theory, however, which invalidates this assumption. One factor is the reaction of clay minerals to liquids, especially water with a low electrolytic content. Another is the failure of the continuum theory in respect to gases at ordinary pressures.

Factors which are indicated by Equation 3.37 to affect $k_w(S_e)$ include:

(1) effective porosity,

(2) maximum pore size,

(3) pore-size distribution,

(4) effective saturation.

The factors k_s and $T_{1.0}$ probably do not vary over a very wide range, but shape of grains might have an important effect on λ, the pore-size distribution index. The same factors affecting $k_w(S_e)$ also affect $k_{nw}(S_e)$.

Another factor of importance, not accounted for in the capillaric theory, is isotropy. The theory assumes that the media are isotropic so that the pore channels have no preferred orientation and neither k_w nor k_{nw} have directional properties. A related factor is the degree of interconnection of the pore space.

3.7.2. Klinkenberg Effect

When the fluid is a gas at normal pressures, a fluid particle meeting the requirements for the continuum analysis usually does not exist. This is because, such a particle must be small compared to the pore dimensions and large compared to the mean free path of fluid molecules. With pore sizes of the order of magnitude of those in common earth materials, these requirements are incompatible.

The error resulting from the assumption of a continuum (with derivatives of velocity existing at the boundary) is usually small in the case of coarse sands, but huge in the case of silts or clays. In the latter case, the difference in k as compared to k_w even when the latter is measured with a liquid that does not react with clays, may be a factor of 2 or 3.

The reason for this huge discrepancy is that the specific surface for fine materials is large, and the boundary condition of a zero velocity at the solid surface is not satisfied in the case of gases unless the mean pressure is, at least, 5 atmospheres. This phenomenon has been called gas slippage and, in respect to flow in porous media, the Klinkenberg effect.

Klinkenberg (1941) measured gas permeabilities as a function of the inverse of mean pressure in a variety of media. He obtained data typified by that shown in Figure 3.8.

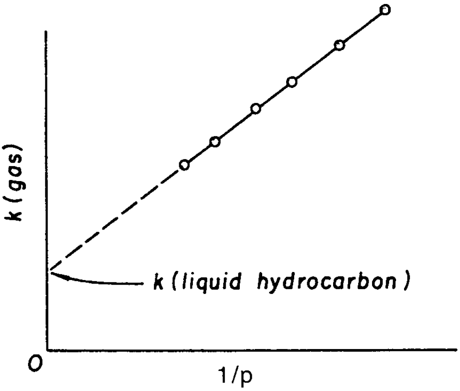

Figure 3.8. Permeability of a gas as a function of the inverse of mean pressure.

Klinkenberg found that the extrapolation of the curve (to $1/p_m$ equal zero) gave a value of k which was identical to that for hydrocarbon liquids. In the petroleum literature the permeability to gas found by this type of extrapolation is sometimes designated by k, without subscripts, and is called simply permeability. It is regarded as being a property of the pore geometry unaffected by reaction with liquids, e.g., clay swelling.

3.8. DARCY EQUATION

Despite the obvious shortcomings of the theory upon which Equation 3.28 is based, it is to be expected that an equation of the form

$$\mathbf{q_i = -\frac{k_i}{\mu}\frac{\partial p^*}{\partial x_i}\delta_{ij}} \qquad 3.40$$

or its equivalent

$$\mathbf{q_i = -\frac{k_i\,(\rho g)}{\mu}\frac{\partial H}{\partial x_j}\delta_{ij}} \qquad 3.41$$

would be a good approximation for a medium containing only one fluid of constant density. Here δ_{ij} is the Kronecker Delta and the subscript i implies that the medium may not be isotropic so that q_i is a component of **q** in a direction i, and k_i is a permeability associated with this direction. In the most general case k should be regarded as a tensor quantity. However, by aligning the coordinate system with the principle axes in respect to permeability, the k tensor is converted to a diagonal tensor and k becomes a vector quantity in effect. It is assumed here that this is always possible.

Equation 3.40 or 3.41 can be regarded as describing a flux component at a point. If the flux rate is integrated over a large area and $\partial p^* / \partial x_i$ is integrated over a macroscopic portion of the flux path, the result is

$$\mathbf{\frac{Q_i}{A} = -K_i\frac{\Delta H}{L_j}\delta_{ij}} \qquad 3.42$$

in which A is an area of a porous system normal to i, K_i is related to k_i by

$$\mathbf{K_i \equiv \frac{k_i\,\rho g}{\mu}},$$

averaged over a length L in the direction i, and ΔH is a piezometric head difference measured between two points a distance L apart. The coefficient K_i has the dimensions of velocity and is called hydraulic conductivity to distinguish it from k_i which has the dimensions of L^2. In some literature, however, K is called permeability and k specific or intrinsic permeability. In any case, K includes the effect of both fluid and medium properties, whereas k theoretically includes only geometric properties of pore space.

If the medium is isotropic, the subscript i can be dropped. Equation 3.42 is simplified for this case as

$$\frac{\mathbf{Q}}{\mathbf{A}} = -\mathbf{K}\frac{\Delta \mathbf{H}}{\mathbf{L}} \qquad 3.43$$

in which A is normal to the direction of L.

Equation 3.43 is called Darcy's law and was discovered by Darcy (1856) experimentally. Other forms of the flux equation for flow in porous media are sometimes called Darcy's law also, but they are actually extensions of Darcy's equation.

3.8.1. Generalized Flux Equation

One version of the flux equation having more general validity than Equation 3.40 is

$$\mathbf{q}_i = \frac{\mathbf{k}_i}{\mu}\left[-\frac{\partial \mathbf{p}}{\partial \mathbf{x}_j} + \rho \mathbf{g}_j\right]\delta_{ij}\ . \qquad 3.44$$

Equation 3.44 is more general than Equation 3.40 because it does not assume the existence of a combined force potential. Furthermore, Equation 3.44 is more convenient to use when dealing with the flow of gases as well as liquids. The reason is that the densities of liquids and gases differ by several orders of magnitude. Consequently, it is more convenient to add equations when the forces are expressed in terms of gravity and gradient of pressure rather than as a gradient of a combined potential.

In this text, therefore, combined potentials are not employed where the problem deals with the simultaneous flow of immiscible fluids. If the media are known to be isotropic and a combined potential exists, it is permissible to write Equation 3.44 in the vector form

$$\mathbf{q} = \frac{\mathbf{k}}{\mu}\left[-\frac{\partial \mathbf{p}^*}{\partial \mathbf{x}_i}\mathbf{e}_i\right] \qquad 3.45$$

in which e is a unit vector, and the repeated subscripts imply a summation over three orthogonal directions. The direction of **q** and the negative pressure gradient correspond only when k is a scalar.

3.8.2. Units Used in Flux Equations

Practically every system of units has, at one time or another, been used in connection with flux equations. Often, the systems used have been inconsistent. The latter are often erroneously called "practical" units. In order to discourage the use of inconsistent and varied units, only consistent units are used in this text. A list of units to be used are given in Table 3.1.

Table 3.1. System of units for describing flow in porous media.

Quantity	Symbol	Unit (cgs)	Unit (SI)
Force	$\mathbf{F}$	dyne (d)	Newton (N) = 10^5d
Mass	M	gram (g)	kilogram (kg) = 10^3 g
Length	L	centimeter (cm)	meter (m) = 10^2 cm
Time	t	second (s)	second (s)
Pressure	p	d/cm^2	Pascal (N/m^2) = 10 d/cm^2
Density	ρ	g/cm^3	kg/m^3 = 10^3 g/cm^3
Scalar gravity	g	980 d/g or cm/s^2	9.8 N/kg
Vector gravity	$\mathbf{g}$	980 d/g or cm/s^2	9.8 N/kg
Permeability	k	cm^2 = 10^2 μ^2	m^2 = 10^{12} μ^2
Conductivity	K	cm/s	m/s = 10^3 cm/s
Viscosity	μ	poises (d-s/cm^2)	N-s/m^2 = 10 poises
Volume flux	$\mathbf{q}$	cm/s	m/s = 10^2 cm/s
Surface tension	σ	d/cm	N/m = 10^3 d/cm

The use of an inconsistent system of units in the petroleum industry is so widespread in the literature that it cannot be ignored completely. In petroleum literature, p is expressed in atmospheres/cm^2, viscosity in centipoise. Other units, except k, are as indicated in Table 3.1. The k that results from this system is inconsistent in that it corresponds to no ordinary unit of length squared. It is called a darcy unit d, having a value of about 0.987 x 10^{-8} cm^2 or about 0.987 microns squared. Another unit of k found in petroleum and geology literature is the millidarcy md, equal to 0.001 d.

3.8.3. Non-homogeneity

When a differential equation is used to describe flux at a point, the question of isotropy arises but not homogeneity. However, when an integrated form of the equation is used, the lack of homogeneity may invalidate conclusions drawn from models that assume homogeneity. This is particularly true in respect to k as a function of saturation when the media are layered [Corey and Rathjens (1956)].

When media are layered, such that the layers are very thin, it is difficult to analyze the individual layers as separate systems. In this case, the assembly of layers act as an anisotropic system. This may drastically alter $k_{nw}(S_e)$ and to a somewhat lesser extent $k_w(S_e)$. Consider, for example, a rock consisting of thin layers with contrasting values of p_e from which samples have been taken using a core drill both parallel and normal to the bedding planes. Sketches of some hypothetical cores of this type are shown in Figure 3.9.

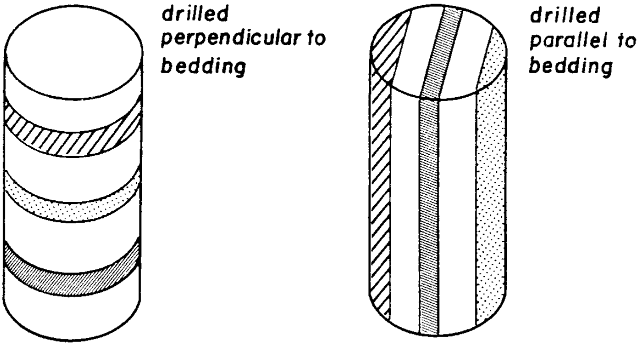

Figure 3.9. Core samples from layered rocks.

The darker layers in Figure 3.9 represent fine materials with a very large p_e. The opposite is depicted as granulated layers. Air permeabilities determined, as a function of average liquid saturation, in an apparatus that permits air flow parallel to the axes of the cores are very different for the two cases and neither resembles $k_{nw}(S_e)$ predicted by Equation 3.34.

In particular, the values of S_m are very small (possibly approaching zero) for the perpendicular core and large (possibly approaching 1.0) for the parallel core. This is in contrast to a average value of S_m for homogeneous cores of about 0.85. The values of $k_w(S_e)$ also are different from predicted values but the difference may not be quite so extreme as for $k_{nw}(S_e)$. If $S_m < 0.85$, strata across the flow path should be expected. If $S_m > 0.85$, strata parallel to the flow path are to be expected. Many other examples of effects from small scale non-homogeneity can be found.

3.8.4. Capillary Pressure-saturation Function During Drainage

The relationships shown in Figures 2.4 and 2.6 for $p_c(S)$ were obtained by measurements made under static conditions. A question arises as to whether $p_c(S)$ would be the same if the measurements are made while S is continuously changing rather than changing by increments. If there is a difference, it would be expected that k(S) would also change, in view of the theory presented in Section 3.6.1.

Topp et al. (1967) and Corey and Brooks (1975) have presented evidence that such a difference actually exists. The difference arises primarily in the range of $p_c < p_e$. In this range of p_c both the $p_c(S)$ and k(S) functions may be dependent on the rate at which saturation changes take place. During a continuous process when an interconnected nonwetting phase does not exist, practically no desaturation takes place, especially if large masses of media are involved that are not close to an exposed boundary. If the media are allowed to remain at a capillary pressure less than the entry pressure for a long time, it may be possible for a gaseous nonwetting phase to evolve from solution or to pass through restricting liquid films by diffusion.

3.8.5. Typical Permeability-saturation Relationships

Typical curves of k(S) for both the wetting or nonwetting phases are shown in Figure 3.10.

The k(S) functions shown in Figure 3.10 are expressed as $k_r(S)$. Here k_r means relative permeability defined as $k_w/k_{w(max)}$ or $k_{nw}/k_{nw(max)}$, in which the subscript (max) refers to the maximum value of k for the particular phase under consideration. Usually k_{nw}(max) is somewhat larger than k_w(max) because of clay swelling and gas slippage. Permeabilities determined with a hydrocarbon liquid are usually the same as permeabilities to gas determined by extrapolation to infinite mean pressure. Also, if k_w and k_{nw} are both liquids, e.g., brine and oil, the maximum values of k for both phases may be nearly the same.

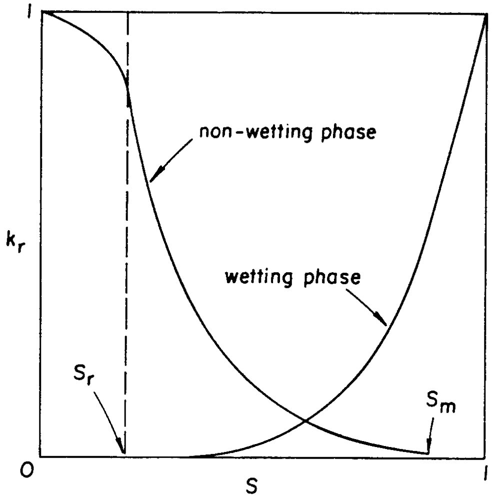

Figure 3.10. Typical relative permeability curves.

Unlike capillary pressure functions, permeability-saturation functions exhibit little hysteresis. However, S_m may be significantly different when obtained on a drainage cycle as compared to S_m on a wetting cycle, although this has not been investigated systematically.

Semi-empirical relationships for k_w (S) have been presented by Brooks and Corey (1964) as described in Section 3.6.1. For the wetting phase $k_{rw}(S)$ is expressed by Equation 3.32, that is,

$$\mathbf{k_{rw} = S_e^{\varepsilon}} .$$

Semi-empirical relationships for $k_w(S)$ have been presented by other authors including:

(1) Childs and Collis-George (1948),

$$\mathbf{K_w = B\theta^3 / s^2} \qquad 3.46$$

in which θ is the volumetric water content, B is a constant, and s is the specific surface.

(2) Irmay (1954),

$$\mathbf{k_{rw} = S_e^3} . \qquad 3.47$$

(3) Corey (1954),

$$\mathbf{k_{rw} = S_e^4 \,.} \qquad 3.48$$

(4) Averjanov (1950),

$$\mathbf{k_{rw} = S_e^{3.5} \,.} \qquad 3.49$$

(5) van Genuchten (1980),

$$\mathbf{k_{rw} = S_e^{1/2}\left[1-\left(1-S_e^{1/m}\right)^m\right]^2 \,,} \qquad 3.50$$

in which m is a dimensionless constant appearing in Equation 2.12.

The relationships of Childs and Collis-George and Irmay could be expressed in the same form, except that the former does not account for the residual saturation. It is doubtful that any relationship that does not account for residual saturation could be valid for anything except sands or soils with very small values of S_r. Irmay's equation corresponds to that of Brooks and Corey for media with a uniform pore size. Experiments by Hausenberg and Zaslavsky (1963), conducted with sands of uniform grain size, confirm Irmay's equation for such materials.

The relationship of Corey (1954) was determined from experiments with a variety of porous rocks that evidently had a pore-size distribution index of about 2, giving an ε value of 4. According to Brooks and Corey (1964), this is a typical value for soil materials as well as porous rocks. Some soils and porous rocks with highly developed structures have values of ε larger than 4. The experiments of Averjanov leading to Equation 3.49 were made with naturally occurring sand deposits. The value of ε obtained by him has been found by Brooks and Corey to be typical for such materials.

3.8.6. Typical k(p_c) Relationships

Typical curves of $k_{rw}(P_c)$ for a fine sandy soil are shown in Figure 3.11, taken from Brooks and Corey(1964). As predicted from Equation 3.30 the curves are linear on a log-log plot over a considerable range of p_c. A discrepancy exists on the drainage cycle mainly in the range of $p_c < p_e$. Although, some decrease in saturation occurs with any increment of capillary pressure greater than zero, k_{rw} curves show no change of k_{rw} over a significant range of p_c.

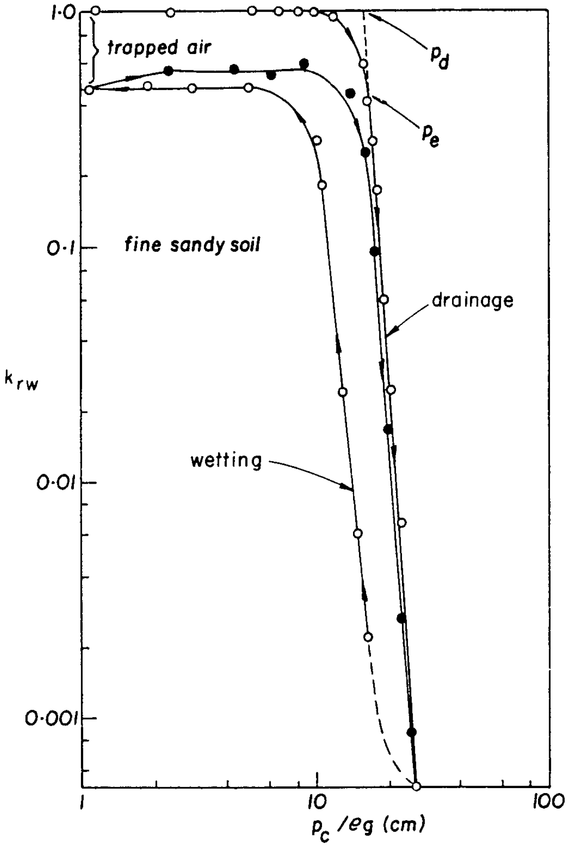

Figure 3.11. Wetting phase relative permeability as a function of capillary pressure on the drainage and wetting cycle.

Evidently, the initial desaturation that occurs at the boundary does not affect the interconnected flow channels. There is some reduction of k_{rw} just before p_e is reached, although this again is apparently a boundary effect. Evidence for the latter conclusion is presented in Figure 3.12, in which a comparison is shown of data obtained by a continuous drainage process with that of a typical curve determined by steady state procedures.

In the case of continuous drainage, k_w is not reduced below 1.0 until the capillary pressure exceeds the entry pressure [Corey and Brooks (1975)]. At this time, liquid pressure may undergo an abrupt but small increase while k_w

continues to decrease rapidly. The abrupt increase in wetting fluid pressure (decrease in capillary pressure) probably results from the sudden entry of air at atmospheric pressure into the interior of the draining column. The value of k_{rw} typically decreases by 50 percent, or slightly more, before p_c again reaches p_e. The capillary pressure continues to be unstable during continuous drainage and to undergo small abrupt decreases as air breaks into new regions of the column. The increase in p_c becomes gradual and stable once p_c exceeds p_e in the entire column.

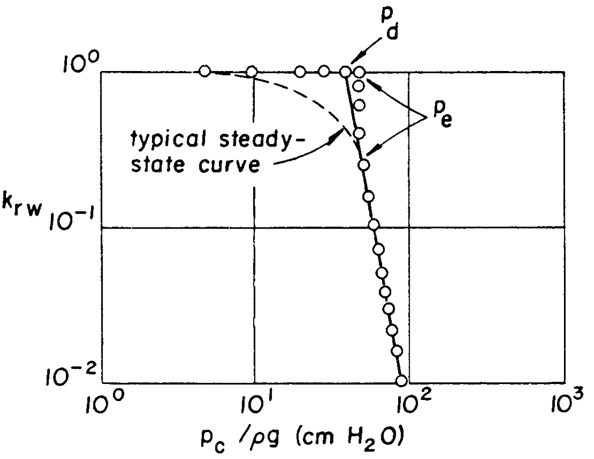

Figure 3.12. Log k_{rw} as a function of log p_c measured during continuous drainage [after Corey and Brooks (1975)].

Air permeability does not exist at capillary pressures less than the entry pressure, as White et al. (1970) have shown. Consequently, it is reasonable to suppose that the erratic behavior of p_c in this range is due to abrupt emptying of certain larger pore channels when air breaks through restricting liquid-filled spaces of somewhat smaller dimensions. Once nearly all of the larger pores contain some air, and this air is interconnected, the unstable behavior stops.

Figure 3.11 illustrates the effect of hysteresis on $k_{rw}(p_c)$. In the case illustrated, the fine sand was first vacuum saturated and then drained by increasing p_c in increments. At a value of p_c of about 2 p_d when k_{rw} had decreased to about 5 x 10^{-4}, p_c was decreased in increments until air was entrapped. At that time the rate of increase of k_{rw} was very slow. A gradual increase in k_{rw} did occur with time, however, without any increase in p_c. This phenomenon has been interpreted by Adams et al. (1969) as being due to a slow diffusion of entrapped air out of the system. Adams et al. found that the diffusion of entrapped air out of a system, in which air permeability does not exist, occurs faster from fine materials than from coarser materials.

When the capillary pressure was again increased with most of the entrapped air still in the soil, k_{rw} did not change until the entry pressure was again reached [see Figure 3.11]. The curve then followed practically the original drainage curve rather than following some intermediate curve.

Semi-empirical relationships for $k_w(P_c)$ on the drainage cycle have been proposed by a number of authors including:

(1) Gardner (1958),

$$\mathbf{k_w = a/\left(b + p_c^n\right)} \qquad 3.51$$

in which a, b and n are constants with dimensions, and values depending on the units employed.

(2) Gardner (1958),

$$\mathbf{k_{rw} = \exp\left(-ap_c\right)} \qquad 3.52$$

in which a is a constant with values depending on the units.

(3) Arbhabhirama and Kridakorn (1968),

$$\mathbf{k_{rw} = 1/\left[\left(p_c/p_d\right)^n + 1\right]} \qquad 3.53$$

in which n is a dimensionless constant.

(4) Brooks and Corey (1964),

$$\mathbf{k_{rw} = 1} \quad \text{for} \quad \mathbf{p_c \le p_d}$$

and

$$\mathbf{k_{rw} = \left(p_d/p_c\right)^{\eta}} \quad \text{for} \quad \mathbf{p_c \ge p_d} \qquad 3.54$$

(5) van Genuchten (1980),

$$\mathbf{k_{rw} = \frac{\left[1-(\alpha h)^{n-1}\left[1 + (\alpha h)^n\right]^{-m}\right]^2}{\left[1+(\alpha h)^n\right]^{m/2}}}, \qquad 3.55$$

in which the symbols are the same as for Equations 2.12 and 3.50.

Equation 3.55 was obtained by substituting Equation 2.12 into an expression derived by Mualem (1976). In order to obtain an exact solution with an equation of reasonable complexity, it was necessary to specify a fixed relationship between the constants m and n, that is,

$$\mathbf{m = 1 - 1/n.}$$

This relationship between m and n implies a capillary pressure-saturation curve with saturation decreasing in the low capillary pressure range to a greater extent than measurements show [see Figure 3.11]. The result is calculated permeabilities decrease in the low capillary pressure range to a much greater extent than experiments indicate.

It is impossible, with the restrictions placed on m and n, for Equation 3.55 to approximate a portion of a curve where experiments show permeability to be invariant with capillary pressure. However, the smooth function is convenient for some types of computation. For this reason, the van Genuchten equation has gained wide popularity.

The Brooks-Corey equations are unique in that they express the relationship between capillary pressure and permeability with two separate functions rather than as a single continuous function. This is in accord with experimental observations that indicate k_w remains invariant over a significant range of p_c.

Equation 3.52 provides the poorest fit to actual data, but it is the easiest to incorporate into mathematical solutions. Consequently, it is the favorite function among many applied mathematicians. Neither Equation 3.51 nor Equation 3.52 are dimensionally consistent so the constants used depend on the units employed. Equation 3.53 is much like Equation 3.51 except that the former is dimensionally consistent.

Brooks and Corey (1964) also have published data for $k_{rnw}(p_c)$. Typical curves of this type are shown in Figure 3.13 for several media representing a wide range of pore-size distributions. The smooth curves shown in Figure 3.13 were calculated by combining Equations 2.9 and 3.34, that is,

$$\mathbf{k_{rnw} = \left[1 - \left(\frac{p_d}{p_c}\right)^{\lambda}\right]^2 \left[1 - \left(\frac{p_d}{p_c}\right)^{2+\lambda}\right]} \qquad 3.56$$

for $\mathbf{p_c \geq p_e}$.

The agreement between theory and measured data is very good considering the highly divergent nature of the materials studied. The materials did have one important property in common. All samples used to obtain the data in Figure 3.13 were relatively homogeneous and isotropic.

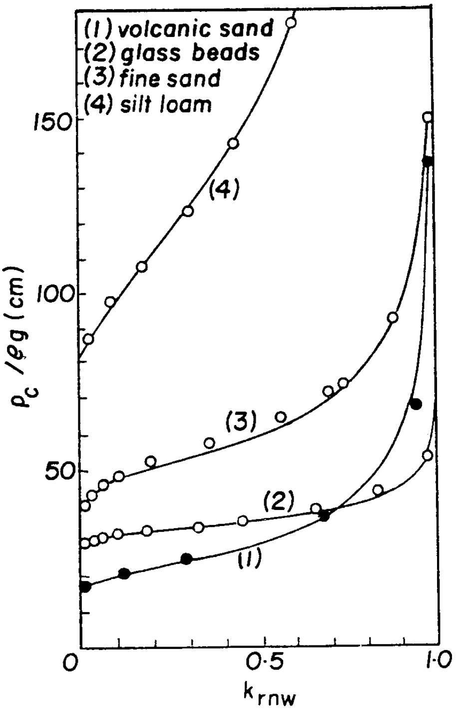

Figure 3.13. Relative permeability of air as a function of capillary pressure compared with theoretical function.

It is essential to remember that Equation 3.56 holds only for capillary pressures greater than the entry pressure and saturations less than S_m. Since p_e is somewhat larger than p_d, a small but significant nonwetting fluid permeability is calculated for the saturation S_m. The predicted finite nonwetting fluid permeability at first breakthrough (of the nonwetting fluid) agrees with the experimental observations of Brooks and Corey (1966) and White, et al (1970). An exact value of S_m can be obtained only from experimental measurements. However, a good estimate of S_m for homogeneous and isotropic media is 0.85.

3.9. SOIL-WATER "DIFFUSION"

For the special case where capillary pressure is a single-valued function of wetting-phase content, it is possible to write the flux equation

$$\mathbf{q_{wi} = \frac{k_w}{\mu_w}\left(-\frac{\partial p_w}{\partial x_i} + \rho_w\, g_i\right)}$$

in terms of a gradient of wetting-phase content. Since this procedure is practical only for flux of water in partially saturated soils, the symbols used in this connection are those employed by soil scientists. That is, q_i is a flux of water in the i direction, K is the conductivity of water, $k_w \rho_w g / \mu_w$, and is assumed to be a scalar. The negative gauge pressure head of water is designated as ψ; that is, ψ is $-p_w/(\rho_w g)$

With these symbols, the flux equation is written as

$$\mathbf{q_i = K\left(\frac{\partial \psi}{\partial x_i} - \frac{\partial h}{\partial x_i}\right)}$$

Assuming ψ to be a single-valued function of θ permits a transformation of variables such that

$$\boldsymbol{\frac{\partial \psi}{\partial x_x} \equiv \frac{d\psi}{d\theta}\frac{\partial \theta}{\partial x_i}},$$

where θ is ϕ S.

A function $D(\theta)$, also assumed to be single-valued, is defined as

$$\boldsymbol{D(\theta) \equiv -K(\theta)\,\frac{d\psi}{d\theta}},$$

so that

$$\boldsymbol{q_i = -D(\theta)\,\frac{\partial \theta}{\partial x_i} - K(\theta)\frac{\partial h}{\partial x_i}}\ . \qquad 3.57$$

Equation 3.57 was first derived by L. A. Richards (1931). Prior to that time, an analogous equation without the gravity term, was employed on a heuristic basis in recognition of the apparent tendency of water to spread from wetter to drier soil.

Writing the flux equation in the form of Equation 3.57, is a mathematical artifice that results in a loss of generality in the following respects:

(1) The equation is valid only for homogeneous soils such that $\psi(\theta)$ is single-valued.

(2) The air pressure must be atmospheric everywhere, or at least constant, so that $p_w/(\rho_w g)$ is $-\psi$. Therefore, Equation 3.57 cannot be used to describe water-air flow unless resistance to air flow is negligible.

(3) An interconnected air phase must exist at all points in the region considered, otherwise, $\psi(\theta)$ is not single-valued.

Practical problems ruled out as applications of Equation 3.57, or at least made inconvenient to solve, include:

(1) flow in layered, or otherwise non-homogeneous soils,

(2) 2-phase flow with air resistance; for example, it is impossible to analyze what happens as a gas cap breaks into an oil well or what happens when water is driven from a tube of wet soil by air pressure,

(3) infiltration from a ponded source of water, such as, a river bed, canal, reservoir, or even an irrigation furrow,

(4) drainage from a soil immediately above a water table.

Equation 3.57 does have some mathematical advantages, especially for cases of horizontal flow, or when the gravity term is negligible. In such cases, the equation takes the form of a true diffusion equation or a heat-flow equation, for which solutions to many boundary value problems have been tabulated.

The function $D(\theta)$ can also be formulated in terms of the pore-size distribution index of Brooks and Corey as

$$\mathbf{D(\theta_e)} = \left[\frac{\mathbf{K_m \psi_d}}{\lambda \phi_e}\right] \theta_e^{(2+1/\lambda)} \qquad 3.58$$

in which

$$\boldsymbol{\theta_e \equiv \frac{\theta - \theta_r}{\theta_m - \theta_r}} ,$$

$$\boldsymbol{\psi_d \equiv \frac{p_d}{\rho_w g}} ,$$

$$\boldsymbol{\phi_e \equiv \phi_m - \phi_r} ,$$

and

$$\mathbf{K_m \equiv K \ at \ \theta_m} ,$$

the maximum water content.

3.10. MEASUREMENT OF RELATIVE PERMEABILITY

Methods of measuring permeability of a fully saturated porous sample are discussed in texts on soil mechanics and groundwater hydraulics and are not reviewed here. Measurement of relative permeability as a function of either capillary pressure or saturation is a more complex problem, but is fundamental to the development of the mechanics of immiscible fluids in porous media.

An extensive review of methods used by soil scientists has been given by Klute (1972). A review of a few of the more common procedures used to determine $k_r(S)$ or $k_r(p_c)$ on rock cores has been presented by Richardson (1961). The methods can be classified as either steady-state or unsteady-state procedures, but there are innumerable variations of each. Only a few of the simpler procedures are reviewed here. Some procedures can be understood only with additional background theory relative to steady and unsteady flow. A further discussion of such methods is presented in Chapters 4 and 5 as applications of the theory presented there.

3.10.1. Steady-state Methods

The variables: pressure, saturation and flux rate are held constant with respect to time, at particular values of pressure and saturation, until the measurement of permeabilities to both phases is completed. In some cases the permeability of only one phase is determined, but in other cases the permeabilities of both phases are measured for a particular capillary pressure or saturation.

With some procedures, capillary pressure and saturation are permitted to vary along a column and permeability is determined at various points along the column. More often, a uniform capillary pressure or saturation is established throughout a sample or, at least, within a test section of a column during the determination of particular values of the permeabilities. Methods of achieving a uniform capillary pressure and saturation within a sample include:

(1) Downward flow of a liquid wetting phase is initiated under a gravitational force only. The nonwetting phase is air at atmospheric pressure, but air permeability is not determined. The first measurements of $k_w(p_c)$ (of which there is a record) were made using this system by L. A. Richards (1931).

(2) Simultaneous flow of two fluid phases is produced under the same pressure gradient. This procedure was introduced in the petroleum industry by Hassler et al. (1944), for determination of permeability-saturation curves for wetting and nonwetting phases on small cores of porous rock. The method was adapted for use on soil samples by Brooks and Corey (1966).

(3) Two fluid phases are injected simultaneously through a porous sample, with the individual flow rates controlled independently, rather than the pressure gradients being controlled. This is called the Penn State method in the petroleum industry, because it was developed in the Petroleum Engineering Department at Pennsylvania State University.

(4) Upward flow of air through a sample is induced by a pressure gradient equal to the pressure gradient in a static liquid phase. This method was designated as the stationary liquid method by Osoba et al. (1951). It is useful for the determination of air permeability-saturation curves only, but it is very convenient and precise for such measurements on small rock cores.

3.10.2. Richards Method

An apparatus similar in principle to that invented by Richards (1931) is shown in Figure 3.l4. By adjusting the pressure at the upper and lower

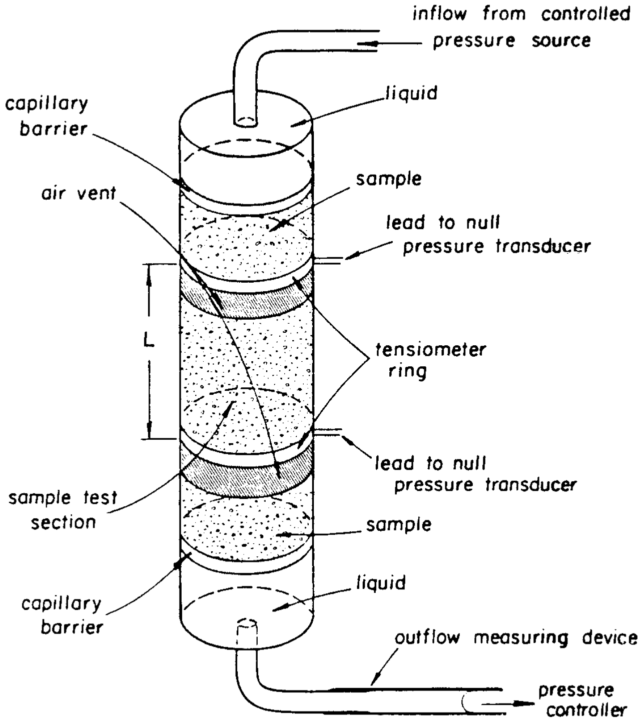

Figure 3.14. Richards permeameter.

barriers to compensate for the head loss through the barriers (and contact regions), the pressure difference in the sample, as recorded over the length L, can be adjusted to zero. Under this condition the wetting liquid flows under the influence of gravity only, and the nonwetting phase (in this case air) is everywhere at atmospheric pressure. In order to be sure that the air is actually at atmospheric pressure, air vents should be provided that do not permit significant evaporation of liquid from the sample.

The outflow rate is measured at the time the pressures at the upper and lower tensiometer rings are recorded, so that Darcy's equation can be solved over the length L. The tensiometers are at some distance inside the ends of the sample to avoid end effects. The tensiometers are designed to contact the circumference of the sample so that the normal flow path through the sample is not distorted.

Pressures at the top and bottom barriers are changed in increments, maintaining a zero pressure gradient at each increment until a steady state is achieved at which time a value of permeability is measured. The corresponding value of capillary pressure is recorded as the difference between atmospheric and liquid pressures, the latter being less than atmospheric after desaturation occurs. The test section of the sample can be scanned with gamma attenuation to determine its saturation also.

Capillary barriers for particular samples, should be selected from materials that have as much permeability as possible, consistent with the requirement that the barrier entry pressure not be exceeded while desaturating the sample. It is not practical to use the same kind of barrier material for all types of samples. When the barriers have a high resistance compared to that of a sample, it is difficult to exercise control over the pressure gradient in the sample. It is also important to use clean liquids, free of algae and other micro-organisms to minimize plugging of the capillary barriers.

With an appropriate change in design, the Richards apparatus can be adapted for use with both rock cores and undisturbed soil samples [Laliberte and Corey (1967)]. It is much easier, however, to obtain precisely reproducible results using hydrocarbon liquids, e.g., soltrol, than with water as the wetting fluid. This is because hydrocarbons have more consistent wetting properties and surface tensions, and are less likely to become contaminated with algae or to cause progressive changes in media structure due to clay dispersion.

3.10.3. Long and Short Columns

One can obtain a more or less uniform capillary pressure and saturation in the upper part of a long column during steady downward flow to a water table located near the base of the column. This can be done without any capillary barriers at the top or bottom and without any adjustment other than that necessary to control the inflow rate at the top of the column.

The long-column technique was first suggested by Childs and Collis-George (1950). It is useful for unconsolidated materials that can be packed uniformly in a long column. Non-homogeneous media prevent the establishment of a uniform pressure in the column. A uniform pressure and saturation in a homogeneous medium can be established in a much shorter column by creating a high capillary pressure at its base rather than maintaining a zero capillary pressure at that point.

The distribution of capillary pressure during steady downward flow in porous media is considered again in Chapter 4 as an application of steady flow theory. Some of the pitfalls in the operation of both long and short column permeameters are discussed in the context of theory presented there.

3.10.4. Air Relative Permeability Measurements

By far, the simplest method of determining air permeability as a function of liquid saturation is the stationary-liquid procedure. With this method air flow is upward through a confined sample of porous medium containing a static liquid. The pressure gradient producing air flow is made exactly equal to the pressure gradient caused by the weight of the static liquid so that a uniform capillary pressure and saturation are established during the measurements of air permeabilities. The corresponding values of saturation are obtained by weighing the sample before and after each permeability measurement. Apparatus used for this purpose is illustrated in Figures 3.15, 3.16 and 3.17.

With this apparatus, a cylindrical sample is confined laterally by a rubber sleeve inside an outer cylinder. When the sample is in place, the sleeve is pressed against the sample by air pressure. If the sample is a rigid rock core,

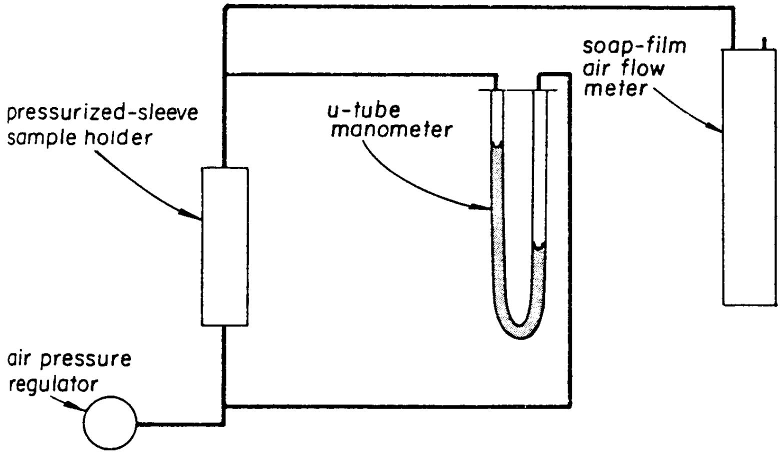

Figure 3.15. Air permeameter with pressurized-sleeve sample holder and soap-film air flow meter.

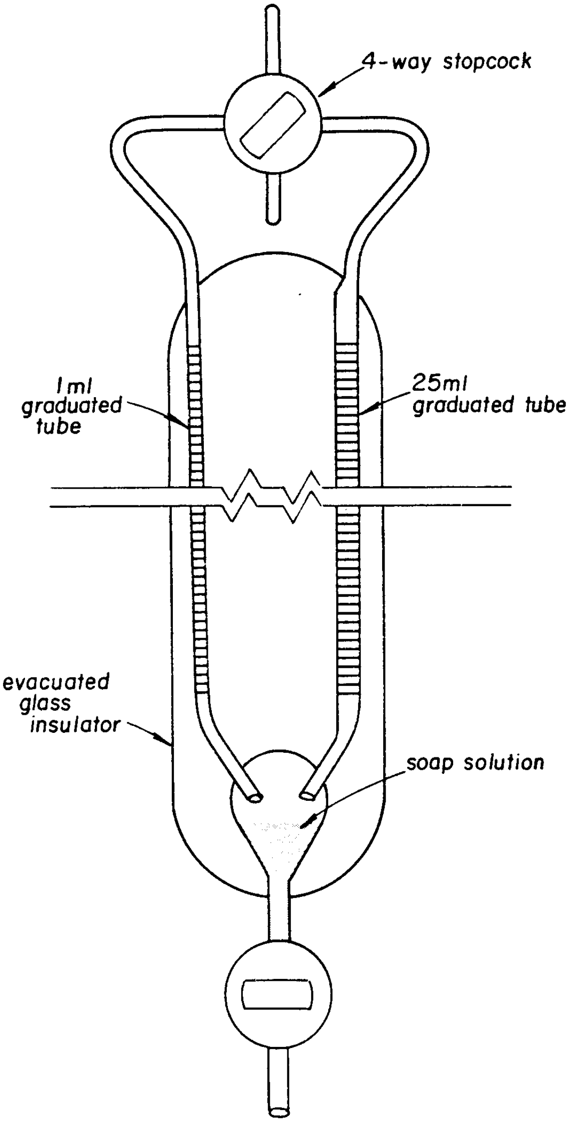

Figure 3.16. Soap-film flow meter.

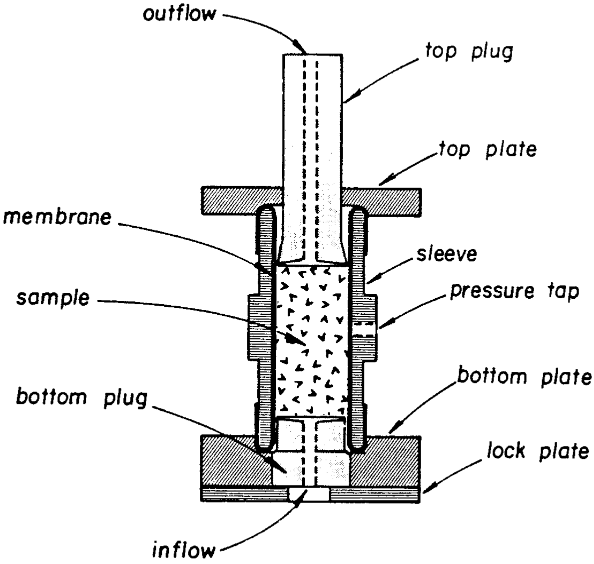

Figure 3.17. Pressurized sleeve sample holder.

the pressure should be about 7 atmospheres. If an unconsolidated sample is used, a pressure of 1/3 atmosphere is sometimes sufficient to prevent bypassing of air, but even this pressure often causes compression of a wet sample and a reduction in air permeability, thus causing the measurements to be erratic.

The top and bottom of the sample must be open to air flow, and in the case of unconsolidated samples, this requires some type of low resistance screen to support the sample at the bottom. Provision must be made to change the liquid saturation between measurements. With rock samples, this is easily done by blotting the sample with an absorbent tissue or simply allowing the liquid to evaporate until a desired weight is obtained. In the case of unconsolidated samples, the problem of desaturation is more complex, and a number of procedures have been devised. Brooks and Corey (1966), for example, employed a sample that was packed into an annular space between an outer pressurized sleeve and a hollow porous cylinder through which the liquid could be extracted. The saturation was computed by a material balance procedure.

The stationary liquid procedure requires a method for measuring very small flux rates precisely. The flux rates cannot be varied by changing the pressure difference causing flow because the air pressure gradient must be held equal to the static pressure gradient in the liquid, in order to maintain a uniform capillary pressure and saturation in the sample. For small cores of about 2.54 cm diameter, air flows are best measured with a soap-film flow meter with at least two calibrated tubes of varying diameters. The smaller tube should permit the precise measurement of flow rates at least as small as 0.01 ml/s. A soap-film tube made from a 1-ml pipette is satisfactory for the small flow rates whereas a 25 ml or larger pipette is useful for the larger flow rates.

3.10.5. Unsteady-state Methods

No attempt is made to induce a uniform saturation within a test section of a sample with an unsteady-state method. The pressure, saturation, and flux rates are permitted to vary with time at all points within a sample. The computation of relative permeability from measurements made during such experiments are usually indirect and depend upon the solution of unsteady flow equations.

The best known procedure under this classification is the Welge (1952) technique that depends upon an integration of the Buckley-Leverett displacement equation. With this method, a wetting fluid is displaced from a sample by a nonwetting fluid under an extremely large pressure gradient. The pressure gradient is large enough to permit the assumption that the two fluids flow through the sample under the same pressure gradient, even though the saturation is nonuniform. Values of permeability are determined from the instantaneous rates at which the two fluids emerge from the outflow end of the sample. The corresponding values of saturation are those calculated for the outflow end of the sample by the Welge integration. This subject is discussed in Chapter 5 as an application of unsteady flow theory.

Other unsteady procedures include variations of the outflow method, first described by Gardner (1956). A saturated soil sample is placed in a capillary pressure cell and desaturated by applying increments of capillary pressure. In the original version, outflow was in response to very small increments. Doering (1965) modified the procedure by desaturating samples in one-step increments. In any case, values of soil water diffusivity D are obtained indirectly by applying unsteady flow theory. Permeability values are not measured directly.

Outflow methods have not given consistently reproducible results because the resistance of the region of contact between samples and capillary barriers varies erratically with the capillary pressure applied.

Attempts have been made to account for this resistance by making the assumption that resistance across the barrier is proportional to the flux rate, but this assumption is not even close to being satisfied, because of the varying resistance at the barrier boundary. Air diffusing through the capillary barrier and reappearing as bubbles in the outflow system also causes problems.

A procedure called the continuous drainage method by Corey and Brooks (1975) is an exception among the unsteady-state procedures in that more or less uniform (although continuously decreasing) saturation is produced in a portion of a draining column where observations of capillary pressure and saturation are made. Like other column procedures, downward flow is produced under the influence of gravity only. This procedure might also be called a "pseudo-steady" method, because the pressure distribution in the column is nearly like that in a column during steady downward flow. The theory of this behavior is also discussed in Chapter 4.

3.11. NON-DARCY FLOW

From theoretical considerations leading to the generalized Kozeny-Carman equation, it is expected that a linear flux equation would be invalid for velocities such that fluid inertia becomes a factor. Although the convective acceleration term, in the Navier-Stokes equation of fluid motion, should be statistically zero when integrated over a macroscopic area and distance, the local effect of such acceleration may be to create a spatial distribution of velocity different from that predicted by ignoring inertia. In this case, the shear force increase may be non-linear with respect to the average flux rate. This causes a corresponding non-linear increase in piezometric gradient. However, the non-linear response is not often associated with turbulence in porous media.

By analogy with flow through large conduits, a Reynolds number Re is often used as a criterion to define the regime for which inertial forces are unimportant. According to this analogy

$$\mathbf{Re} \equiv \frac{\mathbf{gD\rho}}{\boldsymbol{\mu}} ,$$

in which D is some length parameter characterizing the pore geometry. It would appear that D should be related to the largest pore size. Perhaps σ/p_d would be a reasonable choice, but innumerable parameters having the dimension of length have been suggested by various authors. Among these are:

(1) Collins (1961), $\mathbf{D} \equiv (\mathbf{k}/\phi)^{1/2}$,

(2) Ward (1964), $\mathbf{D} \equiv \mathbf{k}^{1/2}$,

but by far the most often used is some D determined by a sieve analysis. For example, the I0 percent size or the 50 percent size.

In practically all cases when Re is based on a grain diameter, Darcy's equation is valid unless Re exceeds an Re between 1 and 10. This removes the problem of inertia effects from the type of applications considered in this text. However, if non-Darcian flow of this type is suspected, it can be verified by plotting the flux rate as a function of the piezometric gradient. If the flow is Darcian, the data should plot as a straight line. If the flux rate deviates toward lower values than a linear relationship indicates, inertia effects may be suspected.

A variety of equations have been proposed to describe non-linear flow of the latter type. One of the first, proposed by Forchheimer (1901), is of the form

$$\frac{\partial \mathbf{p}^*}{\partial \mathbf{x}_i} = \mathbf{a}\mathbf{q}_i + \mathbf{b}\mathbf{q}_i^2$$

in which a and b are constants. A verification of an equation of this form has been presented by Ahmad and Sunada (1969). When equations of this form are used, the constant b is small so when the flux rate is small, the equation reduces to Darcy's law.

Another class of non-Darcian flow may occur at very small velocities. It has been theorized that such flows could result from one of several causes, such as:

(1) adsorptive force fields near solid surfaces that interfere with the normal Newtonian behavior of water,

(2) electrostatic force fields set up in the boundary layer as a result of flow tangential to the boundary (i.e., streaming potentials) which act to oppose the flow,

(3) diffusion or other molecular transport processes acting concurrently with convection and responding to a potential gradient differently from the response of convection.

Many cases of so-called non-Darcian flow reported in the literature are undoubtedly due to faulty laboratory techniques or misinterpretation of the measurements made. One example, is the use of a piezometric potential for cases in which the density is not constant, or the use of some other so-called total potential in an attempt to describe transport resulting from a variety of unrelated mechanisms [Corey and Kemper (1961)].

A review of non-Darcian flow, including a discussion of experimental pitfalls leading to misinterpretation of observed behavior, has been presented by Kutilek (1972). Other authors who have written extensively on the subject of non-Darcian flow include Swartzendruber (1962), and Bolt and Groenvelt (1969).

3.12. POTENTIAL FLOW

In groundwater systems (and some petroleum reservoir situations), it may be possible to regard the flow system, on a macroscopic basis, as being irrotational. This requires the aquifer to be homogeneous, isotropic and to have a constant saturation with a homogeneous fluid. Under such conditions it may be possible to define a velocity potential as the scalar, -KH, and a conjugate function called a stream potential. It may also be feasible to assume divergence of the flow is zero.

If these conditions are satisfied, many flow problems can be solved by the application of potential flow theory. This subject is discussed at length in most books on flow in porous media and groundwater hydraulics. However, for the simultaneous flow of two or more immiscible fluids, the saturation is not necessarily constant so that the divergence is, in general, not zero. Furthermore, the saturation depends on the fluid pressure so potential flow theory is not useful for the applications analyzed in this text.

PROBLEMS AND STUDY QUESTIONS

l. If u_i is an analytic function of the space coordinates even at a solid boundary, what does this imply about the magnitude of u_i at a solid boundary? Explain.

2. If u_i does not vary with time at particular points in a system, what does this imply about the acceleration of fluid particles? Explain.

3. Shear acts throughout a fluid volume. Explain why, in evaluating the force per unit volume acting on a fluid particle due to shear, it is necessary to sum the shearing forces only on the boundaries of the particle.

4. Although the direction of shearing forces on opposite faces of a fluid particle are often in opposite directions, the direction of shear at solid boundaries is always in a direction opposite to u_i of the particle. Explain.

5. Despite the fact that pressure in air varies substantially at different elevations within the Earth's atmosphere, it is often possible for atmospheric scientists to define a combined potential, including pressure and gravitational components, for analyzing motion of air masses. This is called the barotropic case. Explain.

6. It is well known that when two water solutions with different concentrations of salts are separated by a membrane that excludes the salt, water passes through the membrane toward the solution with the higher salt concentration until the pressure is sufficiently higher on that side to counteract the higher osmotic concentration. What can one conclude from this concerning pore size in the membrane relative to the size of fluid particles? What happens if the pore sizes are made substantially larger, say the size of a pin hole? Explain.

7. If one attempts to derive Poiseuille's equation by regarding the entire volume of fluid held in a section of tube as a free body, and balancing shear and pressure forces on this body, what additional assumption is necessary as compared to when forces are balanced on fluid particles?

8. State as simply as possible the mathematical reason why the Kozeny-Carman equation is not valid for porous media having a range of pore sizes.

9. According to the Kozeny-Carman equation, if one were to mix a teaspoonful of montmorillonite with 10 liters of sand, the value of k calculated might be reduced by an order of magnitude. Explain why this is probably unrealistic.

10. Derive an equation for k_{rnw} as a function of p_c and λ from the generalized Kozeny-Carman equation.

11. Would you expect the generalized Kozeny-Carman equation to give better results in terms of relative permeabilities or in terms of permeability per se? Explain.

12. Consider a case of steady downward flow of water through a column of homogeneous soil. The system has the following properties:

    (1) The pressure in the water is uniform.
    (2) The porosity of the soil is 0.5.
    (3) The saturation of the soil is 0.7.
    (4) The residual saturation is 0.3.
    (5) When fully saturated, k_w for the soil is 10^{-8} cm^2.

    If dye is injected into the column at some point, estimate the time for some of this dye to appear at a point one meter lower in the column. What if any effect would molecular, or other dispersion phenomena, have on the estimated time of arrival.

13. Estimate a mean distance that dye would have traveled upon arrival at a point l meter from the starting point. Assume that the dye molecules are too large to be affected significantly by molecular dispersion.

14. The U.S. Salinity Laboratory at Riverside, California once proposed an index of soil-structure stability obtained from the ratio of k_w (when fully saturated with water) to k_g of a dry sample. Explain why such a ratio might not be a measure of stability of soil structure only.

15. Using the generalized Kozeny-Carman equation, derive an expression for k_{rw}, a function of S, for a sand having a completely uniform pore-size and a residual saturation of zero. What would be the theoretical values of ε and λ for this case?

16. A sample of porous medium has a value of k_0 (to oil) of 0.5 μ^2 as measured in a laboratory at sea level. Estimate the value of k_0 (to the same oil) if the measurement is made in a laboratory at a site where the temperature is colder by 10 degrees centigrade and g is 1 percent less. Estimate the conductivity K to an oil with a viscosity of 2.5 centipoise and density of 0.75 gm/cm^2. Would you expect the permeability to water to be equal to, more than, or less than 0.5 square microns? What about the permeability to air?

17. Adams et al. (1969) found that entrapped gas diffuses out of samples of fine-textured porous materials (in contact with a source of wetting liquid) faster than from coarser-textured porous materials. Give a theory to explain this behavior.

18. Oil is often forced out of porous rocks by the injection of brine. Explain why a flux equation in the form of Equation 3.57 is not suitable for analyzing the penetration of brine into the rock in this case.

19. Consider a case in which a porous rock sample containing water at a saturation of 0.72 is placed in a permeameter (of the type shown in Figure 3.15) to measure k_g. If the sample is 5 cm in length and the U-tube manometer contains oil with a specific gravity of 0.75, what should be the Δh reading on the manometer to be sure that the brine saturation remains uniform along the length of the sample during the measurement?

20. Describe an experiment to check whether or not a flux equation in the form of Darcy's equation or Forchheimer's equation should be used to describe a particular flow system.

Chapter 4

STEADY FLOW OF IMMISCIBLE FLUIDS

4.1. STEADY FLOW

Steady flow implies that the variables p, q, ρ and S do not vary with time at any point within a system under consideration. When this situation exists, the relationship among the variables can be analyzed by employing equations that do not include time as a variable. However, we first examine the more general case where the variables are not necessarily constant with respect to time and steady flow is considered as a special case of the more general situation.

To analyze the more general case, it is necessary to combine a flux equation with a continuity equation which involves time as a variable.

4.1.1. Continuity for Flow of Two Immiscible Fluids

A continuity equation is one which expresses the conservation of mass for a reference volume element. In some cases an entire system is considered as a reference element. In other cases only a fluid particle is considered. For flow in porous media containing immiscible fluids, it is usually more informative to consider a volume element of the porous medium such as is used to define porosity or saturation at a point.

When a fluid particle is used as a reference element in fluid mechanics, a continuity equation is written as

$$\frac{\partial(\rho v_i)}{\partial x_i} = -\frac{\partial \rho}{\partial t}$$

which states that the divergence of the mass flow is equal to the rate of decrease of density of the fluid. This equation could also be written in respect to a fluid element within a porous medium, but it does not supply the information needed to solve the type of problems considered here.

Such an equation is useful for situations in which the entire flow region remains completely occupied with one fluid. This is not true for a porous medium in the general case. In fact, changes in saturation are usually of much greater significance than changes in fluid density or medium porosity. Consequently, for immiscible fluid systems it is usually permissible to ignore changes in density or medium porosity with respect to time.

Consequently, a continuity equation is written for flow of a particular fluid phase in a two fluid system as

$$\frac{\partial q_{wi}}{\partial x_i} = -\phi \frac{\partial S}{\partial t} ,$$

or

$$\frac{\partial q_{wi}}{\partial x_i} = -\frac{\partial \theta}{\partial t}, \qquad 4.1$$

for which the reference element is a volume element of porous medium. Equation 4.1 is written for the flux of the wetting phase, but it could be written in respect to q_{nw}, because where compressibility is not a factor

$$\frac{\partial q_{wi}}{\partial x_i} = -\frac{\partial q_{nwi}}{\partial x_i}.$$

4.1.2. Simultaneous Flow of Two Immiscible Fluids

Flow equations are a result of combining flux equations with continuity equations. For two immiscible fluids, the equations are written for each fluid separately and then combined. The assumptions are:

(1) The fluids are immiscible so $p_c = p_{nw} - p_w$.

(2) The differential form of Darcy's law applies independently to each fluid.

(3) Medium and fluid properties, e.g., ϕ, k, ρ and μ are practically constant in time and space.

(4) Permeability can be treated as a scalar.

(5) Compressibility is not an important factor.

Based on these assumptions:

(1) $$\frac{\partial p_c}{\partial x_i} = \frac{\partial p_{nw}}{\partial x_i} - \frac{\partial p_w}{\partial x_i} ,$$

(2) $$q_{wi} = \frac{k_w}{\mu_w}\left(- \frac{\partial p_w}{\partial x_i} + \rho_w g_i \right) ,$$

(3) $$q_{nwi} = \frac{k_{nw}}{\mu_{nw}}\left(- \frac{\partial p_{nw}}{\partial x_i} + \rho_{nw} g_i \right) ,$$

(4) $$\frac{\partial q_{nwi}}{\partial x_i} = \phi \frac{\partial S}{\partial t} .$$

For the special case of steady flow of two fluids, the first three equations are combined to give

$$\frac{\partial p_c}{\partial x_i} = \left[\frac{q_{wi}\mu_w}{k_w} - \frac{q_{nwi}\mu_{nw}}{k_{nw}} \right] - (\Delta\rho) g_i . \qquad 4.2$$

4.2. STEADY FLOW OF A WETTING FLUID

Equation 4.2 can be rewritten as

$$\frac{\partial p_c}{\partial x_i} = \Delta\left(\frac{q_i \mu}{k} \right) - (\Delta\rho) g_i .$$

in which Δ denotes the quantity evaluated for the wetting fluid minus the quantity evaluated for the nonwetting fluid. Although Equation 4.2 contains the fluxes for both fluids, the pressure for neither fluid appears in the equation explicitly. Consequently, the equation can not be employed to determine the distribution of pressure of either fluid. It is useful only for determining the distribution of capillary pressure.

If only the wetting fluid is flowing, Δ can be dropped from the first term so that

$$\frac{\partial p_c}{\partial x_i} = \frac{q_{wi}\mu_w}{k_w} + (\Delta\rho) g \frac{\partial h}{\partial x_i} . \qquad 4.3$$

Solving Equation 4.3 for q_{wi} gives

$$q_{wi} = \frac{k_w}{\mu_w}\left[\frac{\partial p_c}{\partial x_i} - (\Delta\rho) g \frac{\partial h}{\partial x_i}\right], \qquad 4.4$$

the flux equation for steady flow of a wetting fluid in a two-phase fluid system. Equations 4.3 and 4.4 have been derived assuming the nonwetting fluid is static, so the pressure distribution of the nonwetting fluid is determined if its pressure is known at any particular elevation. In this case, solutions of Equations 4.3 and 4.4 provide information about the distribution of fluid pressures as well as capillary pressures.

4.2.1. Steady Downward Flow of Water Through a Homogeneous Petroleum Reservoir

Accumulations of oil are often found in geological traps through which water is flowing steadily downslope. The trap consists of an aquifer (containing brine) between impermeable strata inclined at some angle with respect to a horizontal plane. At some point higher in the aquifer, a porous rock having an entry pressure higher than the entry pressure of the remainder of the aquifer has blocked the upward migration of oil that otherwise would have occurred due to the lower density of oil compared to the brine originally saturating the entire aquifer. A geological trap of this type is shown in Figure 4.1.

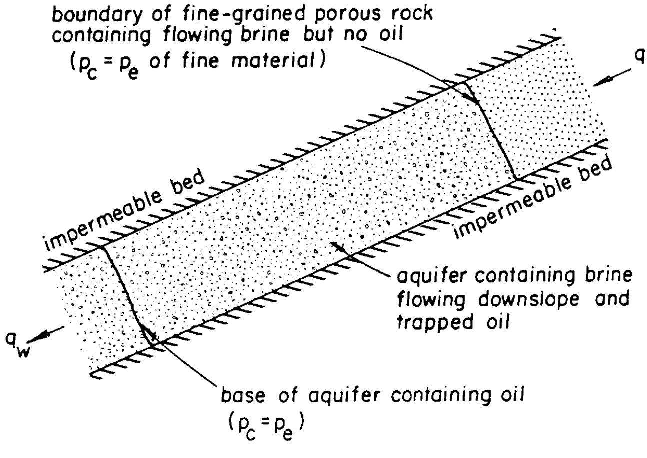

Figure 4.1. Geological oil trap with brine flowing steadily downslope.

Questions to be answered in regard to the oil trap include:

(1) What is the extent of the oil body below the cap rock?

(2) How much oil is in the trap?

(3) How is the oil distributed?

(4) Where is the best place to drill wells to recover the oil?

To answer the questions listed, the distribution of capillary pressure within the oil body must be determined. The maximum capillary pressure (oil into brine) occurs at the cap rock. Capillary pressure cannot be higher than the entry pressure of the cap rock without oil leaking through the cap rock and being lost. Likewise, oil cannot exist as an interconnected phase in the aquifer below an elevation where capillary pressure is equal to entry pressure of the rock containing the oil body.

The distribution of capillary pressure within the oil body can be determined by applying the boundary conditions mentioned above to a solution of Equation 4.3. It is assumed downslope flow is 1-dimensional and the aquifer containing the oil body is homogeneous and has a uniform slope making an angle β with a horizontal plane. Since flow is steady, q_{wi} is a constant. However, k_w is a function of p_c.

For the case under consideration, q_{wi} is a positive flux component whether the flow is upslope or downslope. Sin β and the derivative $\partial h / \partial x_i$ are negative when the flow is downslope. It is more convenient to plot capillary pressure as a function of elevation h rather than x_i. Since x_i is equal to h/sin β, Equation 4.3 becomes

$$\frac{dh_c}{dh} = 1 + \frac{q_w \mu_w}{(\Delta\rho) g k_w \sin\beta} ,$$

where the second term on the right is negative because sin β is negative, and h_c is the capillary pressure head $p_c/(\Delta\rho)g$. Because the flow is steady and 1-dimensional, the flow equation is an ordinary differential equation.

A solution can be obtained numerically if experimental data for $k_w(p_c)$ are available. It can be solved using the Brooks-Corey relationship

$$k_w = k_m \left(\frac{p_d}{p_c}\right)^{\eta}$$

where η is $3\lambda + 2$. At the lower boundary of the interconnected oil body, where p_c is p_e, k_w is approximately 0.5 k_m.

The flow equation for this case is written as

$$\frac{dh_c}{dh} = 1 - a\left(\frac{h_c}{h_d}\right)^{\eta} \qquad 4.5$$

in which

$$a = \frac{-q_w\,\mu_w}{(\Delta\rho)\,g\,k_m \sin\beta}$$

and h_d is $p_d/(\Delta\rho)g$. Equation 4.5 applies only where h_c is equal to or greater than h_e, that is where both fluids are interconnected.

The solution of the flow equation is of the form

$$dh = f(h_c)\,dh_c$$

$$f(h_c) = \left[1 - a\left(\frac{h_c}{h_d}\right)^{\eta}\right]^{-1}$$

so that values of h corresponding to particular values of h_c can be obtained by integration. In particular, the elevation difference over which the oil body may extend can be obtained by integrating Equation 4.6 over a range of h_c from $p_e/(\Delta\rho)g$ of the aquifer material to $p_e/(\Delta\rho)g$ for the cap rock.

The integration is difficult to perform analytically because η is usually greater than 6.0. In any case, the analytical expression for the integral is complex, and it is much easier to use a computer program to integrate the function numerically for any value of η.

Qualitative aspects of the relationship $h_c(h)$ can be determined by inspection of Equation 4.5, noting that q_{wi} is typically a very small number. When h_c is equal to $p_e/(\Delta\rho)g$, $k_w \approx 0.5k_m$, and dh_c/dh is only slightly less than 1.0. Thus the distribution of h_c is practically hydrostatic at the bottom of the interconnected oil body. Furthermore, dh_c/dh theoretically cannot be negative, because this would imply a decreasing h which restores the value of dh_c/dh to zero. Consequently, dh_c/dh is approximately 1.0 at the bottom of the interconnected oil body and approaches zero if the oil body extends over a sufficient depth.

A plot of $h_c(h)$ is shown in Figure 4.2. Note that capillary pressure changes abruptly from a practically hydrostatic relationship with elevation to a capillary pressure that is essentially invariant with elevation. This is because η is usually large, a value of 8 being typical. The larger the value of η, the more abrupt the transition.

A case is considered in which the entry pressure of the cap rock is twice the value of p_d for the aquifer rock and η for the aquifer material is 8. It is desired to determine the value of q_w that permits the oil body to reach a maximum saturation at the cap rock and allows the oil body to extend for an indefinite distance downslope with an invariant oil saturation.

Such a condition exists when dh_c/dh approaches zero and p_c equals p_e of the cap rock. The value of q_w can be determined from Equation 4.5 by setting dh_c/dh equal to zero. In this case, "a" in Equation 4.5 is 2^{-8}, and

$$q_w = -\frac{(\Delta\rho)\,g\,k_m \sin\beta}{2^8\,\mu_w} \quad . \qquad 4.7$$

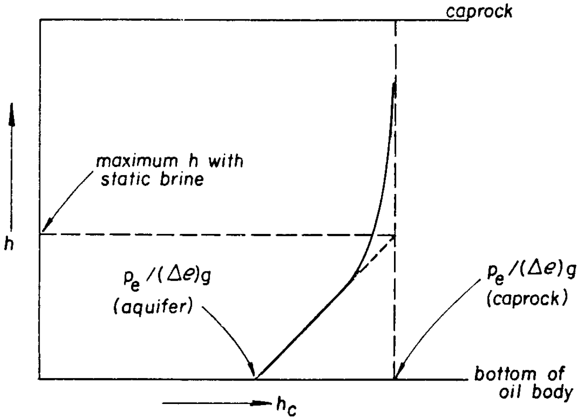

Figure 4.2. Distribution of capillary pressure with elevation in geological oil trap with brine flowing downslope.

A smaller value of q_w results in oil leaking through the cap rock. In the limit as the magnitude of q_w approaches zero, the distribution of capillary pressure is hydrostatic, and the elevation interval occupied by interconnected oil is given by

$$\Delta h = \frac{\Delta p_e}{(\Delta\rho)g}$$

where Δp_e is the difference in entry pressures of the cap rock and the aquifer materials. A value of q_w larger than that given by Equation 4.7 causes the oil body to be stretched over a larger h, but the capillary pressure is smaller than the entry pressure of the cap rock and the oil saturation is smaller.

When the downward flux rate exceeds a critical flux rate q_c given by

$$q_c = -\frac{(\Delta\rho)g k_m \sin\beta}{\mu_w},$$

an oil body cannot exist as an interconnected phase because dh_c/dh is negative. Substituting the critical flux rate into Darcy's equation for the brine shows that the magnitude of a piezometric head gradient that moves all interconnected oil downstream exceeds $(\Delta\rho/\rho_w)\sin\beta$.

Sometimes brine flow is upslope, and in such cases the maximum height of an interconnected oil accumulation is smaller than that given by

$$\Delta h = \frac{\Delta p_e}{(\Delta\rho)g}.$$

In fact, oil bodies have not been observed where flow of brine is upslope although, surprisingly, upslope flows are not uncommon.

In cases where flow is downslope through a geological trap, the most appropriate place to drill oil wells is in the upper part of the oil body near the cap rock. In this region the proportion of brine produced along with the oil is less than elsewhere in the trap. In aquifers that are not homogeneous, distributions of capillary pressure are different from that shown in Figure 4.2. The latter situation is discussed in Section 4.2.5.

4.2.2. Steady Flow of Water Through Aquifers Containing a NAPL

Problems similar in theory to those discussed for petroleum reservoirs are encountered as a result of the introduction of non-aqueous polluting liquids into fresh water aquifers. Non-aqueous polluting liquids are designated as NAPLs. Those less dense than water are called LNAPLs. Oil and gasoline spills, or leaks from storage tanks, are common sources of such pollutants. Liquids more dense than water are known as DNAPLs.

DNAPLs of greatest concern are chlorinated solvents used as cleaning solutions for clothing and machinery. They are commonly used for cleaning metal parts of airplanes and in de-icing solutions, and are frequently introduced into groundwater around airports and defense facilities. Although DNAPLs are less common pollutants than petroleum fluids, they are even more dangerous because they are usually very toxic, and also because they are more dense than water.

Theory presented in this section relating to steady flow of brine through a petroleum reservoir applies equally to flow of fresh water through aquifers that have been polluted with LNAPLs. The steady flow case with a static LNAPL phase is significant in this regard because, as a first step in a clean-up program, one may try to flush out as much LNAPL as possible by establishing a steady flow of water through a polluted aquifer.

Questions to be investigated include:

(1) What is the best way to conduct a water flushing operation, and what flux rate is needed to remove the maximum amount of mobile LNAPL?

(2) How much LNAPL remains in the aquifer after the LNAPL becomes static and no more is removed by flushing?

(3) How is the LNAPL distributed after it reaches a static condition?

Answers to the questions listed above can be found by applying Equations 4.3 or 4.5. Equation 4.5 assumes the Brooks-Corey approximations for the capillary pressure-saturation and capillary pressure-permeability relationships are valid, but it is useful for calculating at least qualitatively how LNAPLs are distributed in an aquifer during steady (1 – dimensional) flow of water.

Qualitative deductions from an examination of Equation 4.5 include:

(1) $\text{Sin}\beta$ is positive in the upslope direction so the critical flux rate upslope is negative. Consequently LNAPL moves upward with any flow rate until only entrapped LNAPL remains. However, a flow rate greater than zero speeds up the process.

(2) When flow is downslope, a water flux rate greater than a critical flux rate $(\Delta\rho)g\,k_m \ \sin\beta/\mu_w$ is needed to remove all mobile LNAPL. In this case k_m is interpreted as the maximum water permeability for an aquifer that may contain entrapped air as well as entrapped LNAPL. The critical flux rate is that producing a zero capillary pressure gradient with an effective water permeability equal to k_m and a static LNAPL. The piezometric gradient required to produce a critical flux rate is given by $(\Delta\rho/\rho_w)\ \sin\beta$.

(3) A downslope flux rate less than critical in a confined homogeneous aquifer may spread LNAPL as an interconnected fluid (with a uniform saturation) over an indefinite distance in the direction of flow. The saturation in the LNAPL body depends on the medium properties and the flux rate. The length of the interconnected LNAPL body depends also on the volume of LNAPL originally present relative to the total pore volume of the aquifer per unit of length in the direction of flow.

The distribution of capillary pressure and saturation for 1-dimensional flow systems can be estimated from Equation 4.6. The calculation can be made by evaluating the integrals numerically between appropriate limits. However, if the flow diverges or converges due to a changing cross-section for the flow, q_{wi} varies with the space coordinates and the solution is more complex. Another complication occurs where there is a change in the angle β, as for a case where the slope of an aquifer changes.

Equations used to analyze the distribution of LNAPLs apply also to DNAPLs. However, the density difference is negative and often much greater in magnitude than for LNAPLs. Consequently, DNAPLs migrate downward in an aquifer until blocked by a barrier of higher entry pressure than the entry pressure of the aquifer material, or unless they encounter an upward flux of water that exceeds the critical flux rate.

The forgoing theory applies only to convection of water when the nonwetting fluid is static. It is important to keep in mind that other processes including diffusion, partitioning between phases, adsorption of constituents on solid particles, and biodegradation of organic compounds occur simultaneously with convection.

4.2.3. Downward Flow of Water Through a Homogeneous Soil Profile to a Water Table

Equation 4.5 applies also to steady 1-dimensional downward flow through a soil in which air is present as a nonwetting phase. For this case, it is feasible to assume that both p_{nw} and ρ_{nw} are equal to zero, and sin β is -1.0, so that

$$\frac{\mathbf{dh_c}}{\mathbf{dh}} = \mathbf{1} - \frac{\mathbf{q_w}}{\mathbf{K_m}}\left(\frac{\mathbf{h_c}}{\mathbf{h_d}}\right)^{\eta}$$

for $h_c > h_e$. It is often convenient to scale the variables h_c and h by dividing each by h_d. The scaled version of h_c is designated as $\hat{h}_c$ The ratio q_w/K_m is also a scaled variable designated as $\hat{q}$. The result in terms of scaled variables is

$$\frac{\mathbf{d\hat{h}_c}}{\mathbf{d\hat{h}}} = \mathbf{1} - \hat{\mathbf{q}}\,\hat{\mathbf{h}}_c^{\eta} \; . \qquad 4.8$$

Equation 4.8 assumes that $\hat{h}_c$ is equal to or greater than $\hat{h}_e$ because the equation applies only where both fluids are interconnected. A solution is obtained by numerical integration [P. R. Corey et al. (1982)].

When flow is steadily downward to a water table, the result is qualitatively like that shown in Figure 4.2 for downward flow through an oil trap. In this case, however, there is no cap rock to set a limit on the value of h_c. Instead, air at atmospheric pressure at the soil surface insures that the air pressure will never become less than atmospheric. When $\hat{q}$ becomes greater than 1.0, dh_c/dh becomes negative and an interconnected air phase does not exist. In the limit, as $\hat{q}$ approaches zero, the distribution of capillary pressure becomes everywhere hydrostatic. Figure 4.3 shows a family of distributions for a range of values of $\hat{q}$.

In Figure 4.3, $\hat{q}_1$ is less than 1.0 and $\hat{h}_c$ is small enough that p_c is everywhere less than p_e, and consequently an interconnected air phase does not exist. When this situation exists, Equation 4.8 does not apply. Instead,

$$\frac{\mathbf{d\hat{h}_c}}{\mathbf{d\hat{h}}} = \mathbf{1} - \hat{\mathbf{q}} \qquad 4.9$$

so that dh_c/dh is a constant. Equation 4.9 is used for all values of $\hat{h}_c$ less than p_e/p_d, a ratio slightly greater than 1.0.

The behavior indicated in Figure 4.3 makes possible the long-column method of determining $k_{rw}(p_c)$. Steady downward flow is established at a range of values of $\hat{q}$ such that the capillary pressure is practically constant

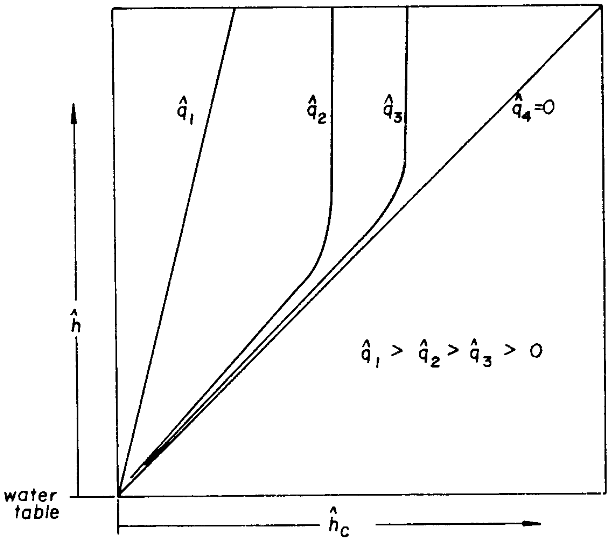

Figure 4.3. Capillary pressure head as a function of elevation during steady downward flow of water through a homogeneous soil.

with elevation over a range of elevations at the top of the column. The value of $\hat{h}_c$ may be determined with tensiometers in the section of the column with an invariant capillary pressure, and saturation can be determined using gamma attenuation. Since pressure is invariant with elevation in this region of the column, $\partial H / \partial z$ is 1.0, and K is equal to q_w. The procedure requires that the column be packed homogeneously, and is sufficiently long that capillary pressure is actually invariant with elevation in the test section.

The length of column needed depends upon the entry pressure of the soil because dh_c/dh is a constant equal to $(1 - \hat{q})$ for values of $\hat{h}_c < 1.0$. Where $\hat{h}_c > 1.0$, the product $\hat{q}\,\hat{h}_c^{\eta}$ increases toward a value 1.0. The transition is short if η is large and somewhat longer if it is relatively small. Consequently, the length of the column needed depends upon η as well as p_e, but the latter is usually the more important factor.

A complication may arise when $K_w(p_c)$ is determined on the drainage cycle, that is, when the flux rate is decreased in increments starting with a fully saturated column. When the flux rate is reduced, a period of time is required before a new steady state exists. Although the flux rate decreases immediately throughout the column, air does not immediately replace the water necessary to produce a new steady state. During the transition period K_w is higher than its steady state value. Consequently, the flux rate is smaller and dh_c/dh is larger than for the steady state intended because the top of the column over desaturates. A steady state is eventually reached, but it is reached from the wetting direction rather than the drainage direction. This produces erratic data for $K_w(p_c)$, but $K_w(S)$ is not affected significantly.

The problem of reversing cycles can be minimized by making the increments of q_w small and providing sufficient air vents along the length of the column to minimize the time for air to penetrate the column in response to a reduction in water pressure.

4.2.4. Steady Downward Flow to a High Capillary Pressure Sink

The column length needed to produce a practically uniform capillary pressure near the top of a column can be substantially reduced by maintaining a high value of capillary pressure at the bottom of a column of homogeneous soil.

In this case Equation 4.8 applies to the entire column provided $\hat{q} < 1.0$. This is because h_c is greater than $p_e/(\Delta\rho)g$ everywhere in the column. The solution is

$$\int f\left(\hat{h}_c\right) d\hat{h}_c = \int d\hat{h}$$

which can be integrated numerically. The integration begins at the lower boundary where the scaled capillary pressure head is at a controlled value greater than 1.0, say 3.0. The integral gives the scaled elevation at which a calculated scaled capillary pressure head occurs.

A curve obtained by this procedure is shown in Figure 4.4 as the solid line. This is compared to an analogous curve (for the same $\hat{q}$ and η) obtained for downward flow to a water table. For steady downward flow to both a high capillary pressure sink and a water table, the value of capillary pressure head in the upper part of the column approaches the same asymptote, providing the scaled flux rate is the same. The value of scaled capillary pressure head at the asymptote depends on the scaled flux rate and the pore-size distribution index. However, with a high capillary pressure sink, the asymptote is approached over a much shorter length of column.

The behavior illustrated in Figure 4.4 permits the determination of $K_w(p_c)$ or $K_w(S)$ using columns shorter than those needed for the long-column procedure. There is an important difficulty, however, in that high

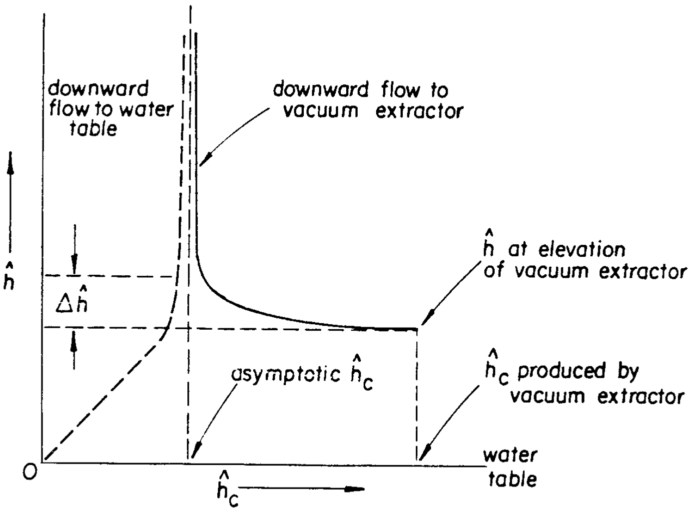

Figure 4.4. Distribution of capillary pressure head during steady downward flow to a high capillary pressure sink compared to flow to a water table. [Adapted from P. R. Corey et al. (1982)].

capillary pressure sinks are not easy to maintain. It would seem that the only requirement would be a capillary barrier at the base of a column connected to a siphon tube with an outlet at a sufficiently low elevation.

Unfortunately, the resistance of a capillary barrier (and especially the contact zone between barrier and the soil column) has a tendency to increase rapidly with the discharge of water passing through the column. What starts out being a high capillary pressure sink may quickly become a barrier to flow. Experience has shown that a capillary barrier consisting of a sequence of unconsolidated granular layers, coarser at the bottom and progressively finer at the top in contact with the soil, is the most satisfactory arrangement. Such a barrier has much less tendency to become plugged or to lose contact with the soil than a rigid or semi-rigid porous membrane. The entry pressure of the finest layer in the barrier should not be larger than necessary to hold the vacuum of the outflow tube. Also, the fine layer should not contain clay or organic material.

Another interesting application of a high capillary pressure sink is for monitoring the soil water solution passing through a soil profile. There are numerous occasions for which a measurement of the quantity and quality of the soil leachate is needed. For example, environmental engineers employ such methods for studying pollution of groundwater from a variety of sources.

The principle problem to be overcome is due to the soil water being at some pressure less than atmospheric. Therefore, to get the water into a collection device, the pressure of water in the device (in contact with soil water) must be lower than the soil water pressure. This must be done without permitting the device to fill with air, thus excluding the soil water. However, the soil water pressure at the entrance to the collector must not be substantially lower than that of the undisturbed soil water because this causes excessive convergence of streamlines and invalidates a quantitative determination of the rate at which leachate passes a particular horizontal plane.

Several investigators have utilized porous ceramic capillary barriers operating under a suction for this purpose. Cole (1968) used this technique to measure rate of water flow through a forest soil. Convergence of streamlines towards capillary barriers was prevented by manual adjustment of the suction to that of the water in the surrounding soil.

Duke, Kruse and Hutchinson (1970) improved on Cole's device by installing an automatic system to adjust the suction in the collector to that in the surrounding soil. It was found, however, that the automatic equipment was difficult to maintain as well as being expensive to install. When suction is applied at the bottom of a buried trough (as shown in Figure 4.5) the

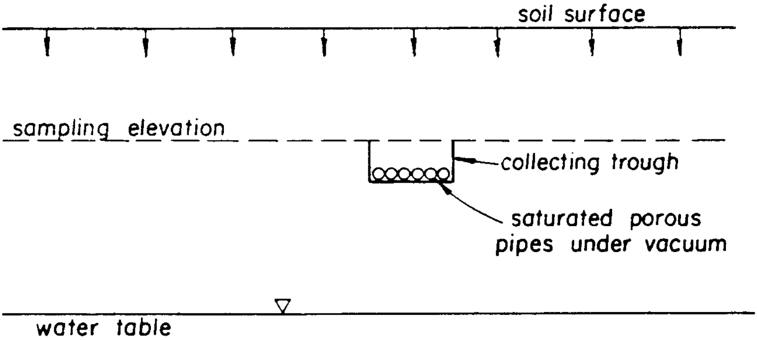

Figure 4.5. Sketch of soil water monitoring system.

amount of leachate collected is insensitive to the amount of suction applied. P. R. Corey et al. (1982) investigated the depth of a trough required to ensure that the suction at the trough would depend only on the vertical component of q_w and would not be affected significantly by the suction at the bottom of the trough.

This problem is tantamount to that of determining the magnitude of $\Delta \hat{h}$ (indicated in Figure 4.4) such that the scaled capillary pressure head is within some arbitrarily small increment of the asymptotic value. When the capillary pressure head is at the asymptotic value, no convergence of flow lines toward the top of the trough should occur. The magnitude of $\Delta \hat{h}$ is greater for soils having small η and for smaller scaled flux rates. The necessary trough height also is affected slightly by the scaled capillary pressure head applied at the bottom of the collecting trough.

P. R. Corey et al (1982) found that a trough 20 cm deep would be satisfactory for typical soil conditions but might not be satisfactory for a soil having a very wide range of pore sizes, say η less than about 6.

4.2.5. Steady Upward Flow from a Water Table

The rate of evaporation from a fallow soil may be controlled by either the capacity of the atmospheric environment to evaporate water or the capacity of the soil to transmit water to the surface. Except where the water table is at very shallow depths, the capacity of fallow soil to transmit water usually is the limiting factor [Anat et al. (1965)].

When upward flow is from a stationary water table, the flux rate may sometimes approach a steady state, and the limiting rate may be determined by application of Equation 4.3. In this case, Equation 4.3 is written as

$$\frac{d\hat{h}_c}{d\hat{h}} = 1 + \hat{q} \qquad \text{for} \qquad \hat{h}_c \leq 1$$

and

$$\frac{d\hat{h}_c}{d\hat{h}} = 1 + \hat{q}\,\hat{h}_c^{\eta} \qquad \text{for} \qquad \hat{h}_c > 1 \quad . \tag{4.10}$$

Equations 4.10 differ from Equation 4.8 in that sin β is positive rather than negative. The value of $d\hat{h}_c / d\hat{h}$, therefore, cannot equal zero anywhere in the soil profile and has a minimum value of 1.0. Figure 4.6 presents a typical solution of Equations 4.10 giving the variation of $\hat{h}_c$ with $\hat{h}$.

Because the change in pressure due to flow and to gravity are additive, the transition from wet to dry soil is very abrupt. For any particular soil (and distance from dry soil to water table) there is a maximum upward flux rate. This fact was first pointed out by Gardner (1958). The maximum upward flow rate (for a given depth from dry soil to water table) occurs when the dry soil is encountered only at the extreme surface, or perhaps when the surface is not completely dry. Further drying causes the upward flow rate to decrease, evidently because of contact angle hysteresis or some such mechanism. The latter phenomenon has been discussed by Schleusener and Corey (1959).

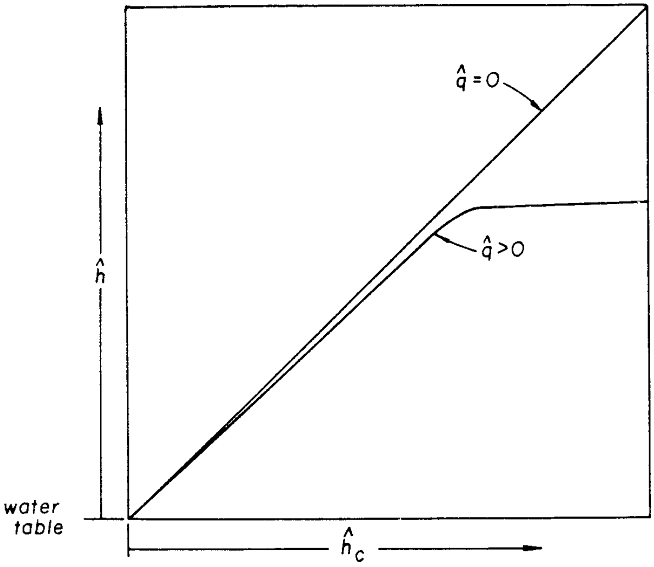

Figure 4.6. Distribution of capillary pressure head during steady upward flow from a water table.

The solution of Equation 4.10 is of the form

$$\int f\left(\hat{h}_c\right) d\hat{h}_c = \int d\hat{h} ,$$

and the integrations are performed numerically. The integration is carried out in two steps. The first step is $\hat{h}_c$ from zero to 1, where $f(\hat{h}_c)$ is a constant equal to $1/(1 + \hat{q})$ and the second step is $\hat{h}_c$ from 1 to ∞, the capillary pressure of the dry soil being assumed to approach infinity. The integration is performed by substituting arbitrary values of $\hat{q}$ and η. The sum of the two integrals gives the scaled depth $\hat{d}$ from dry soil to the water table corresponding to the value of $\hat{q}$ and η selected. By repeating the process for a range of values of $\hat{q}$ and η, H. R. Duke obtained a nomograph [Anat et al. (1965)] showing the relationship among $\hat{q}$, $\hat{d}$ and η.

Anat developed an approximate analytical expression for $\hat{q}(\hat{d}, \eta)$. He expanded $f(\hat{h}_c)$ into a convergent series and integrated term by term. For values of $\hat{q} < 1.0$, including the range for most applications, the relationship is

$$\hat{q} \approx \left[1 + \frac{1.886}{\eta^2 + 1}\right]^{\eta} \hat{d}^{-n} \quad . \qquad 4.11$$

Equation 4.11 is more general, but comparable in form, to equations presented earlier by Gardner (1958) for the special case of integer values of η from 1 to 4. In deriving his equations, Gardner also assumed q was small.

4.2.6. Steady Downward Flow Through Stratified Media

A case is considered in which water is percolating steadily downward through a sequence of layers to a deep water table. The layers have contrasting properties but are homogeneous within themselves. For simplicity, a system is considered in which alternating layers of only two types of media exist. The $K_w(p_c)$ relationships for these two media are shown in Figure 4.7 on log-log plots. The arrangement of the layers is as shown in Figure 4.8.

Continuity requires the flux rate to be the same in all layers since the flow is steady. Furthermore, the water is interconnected (continuous) throughout all layers, including at the boundaries between layers, otherwise there could be no flow. The water pressure, therefore, is continuous at all points in the column because a pressure discontinuity in a continuous liquid phase would imply an infinite driving force at the point of discontinuity in pressure.

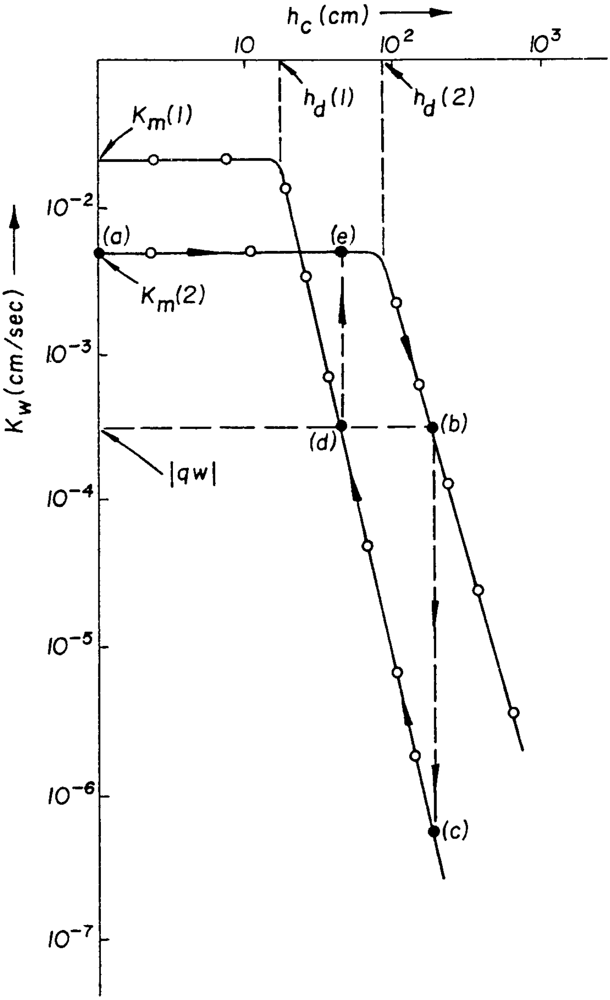

Figure 4.7. K_w as a function of p_c (for two media from a layered soil) on a log-log plot.

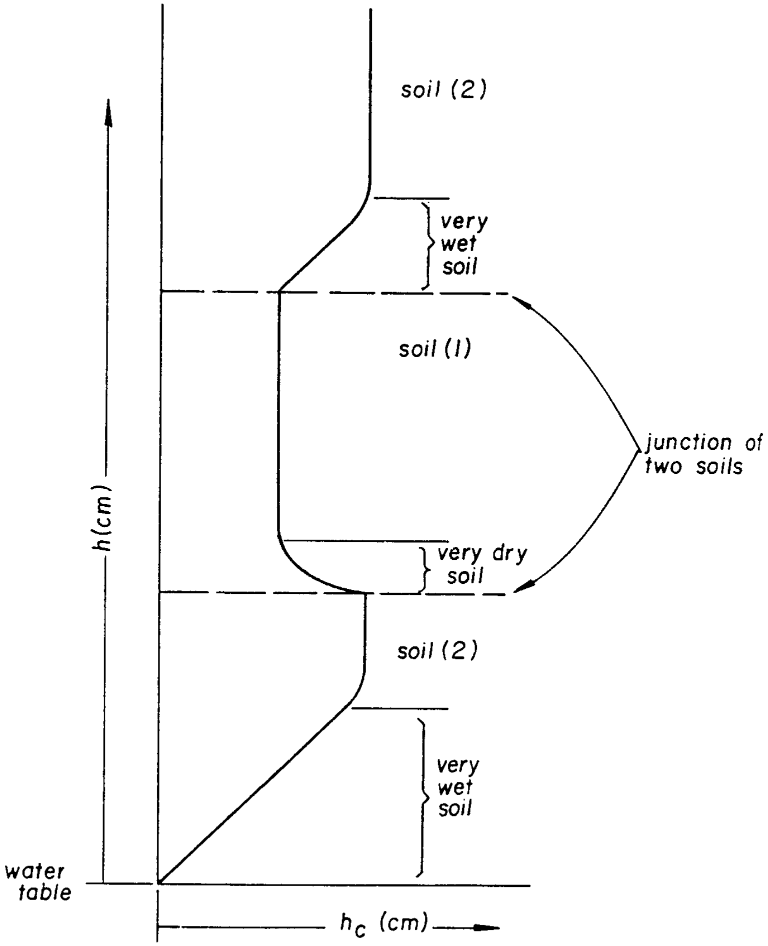

Figure 4.8. Distribution of capillary pressure during steady downward flow of water through layered soils.

The value of dh_c/dh at all points is given by Equation 4.3, written for the purpose of this analysis as

$$\frac{dh_c}{dh} = 1 - \frac{q_w}{K_w}$$

in which K_w is a function of h_c. The solution of the equation is

$$\int f(h_c)\,dh_c = \Delta h \ .$$

When the integration is carried out across boundaries between layers, however, there is an abrupt change in K_w which produces a discontinuity in dh_c/dh. Consequently, h_c is a continuous function of the space coordinates but not an analytic function in this case.

The shape of the curve shown in Figure 4.8 can be understood by reference to the differential equation of flow. When K_w equals q_w, dh_c/dh is zero, the hydraulic gradient is 1.0, and the driving force is entirely gravitational. This situation exists in the regions indicated by the vertical portions of the curve.

It is informative to consider the behavior of dh_c/dh starting at the water table in soil(2). The point (a) on the $K_w(h_c)$ curve for soil(2) indicates the value of h_c and K_w at this point. Immediately above this point, h_c increases practically in a hydrostatic manner because $q_w \ll K_m$ of soil(2). This trend continues until h_c exceeds h_d of soil(2). The curve then bends until h_c reaches a value indicated by point(b) in Figure 4.7, where K_w equals q_w. The curve is then vertical because dh_c/dh is zero. Where soil(1) is encountered, however, K_w is very small, as indicated for point(c). The ratio q_w/K_w is large, and dh_c/dh has a large negative value. This causes h_c to decrease rapidly to a value indicated by point (d) where K_w again is equal to q_w and the curve is again practically vertical. At this value of h_c , however, K_w in soil(2) is as indicated by point(e). Where this soil is encountered, the pressure distribution is practically hydrostatic until h_c (indicated at point(b)) is again reached and the curve again becomes vertical.

As indicated in Figure 4.8, the wettest soil (other than near the water table) is just above the coarse material. The driest soil is in the coarse material just above the underlying fine soil. In fact, the resistance to flow in the thin dry layer on top of the fine soil, is probably greater than the combined resistance of the remainder of the profile. In other words, the flux rate is controlled by this region of the profile. Conditions analogous to this case occur frequently in nature and for this reason, layered soils drain much slower than homogeneous soils.

The behavior illustrated in Figure 4.8 was demonstrated experimentally by Scott and Corey (1961) who were the first to apply Equation 4.3 to flow through layered soils. They showed that Equation 4.3 described the experimentally measured pressure distributions during steady downward flow in layered soils for the cases they investigated.

Equation 4.3, in its more general form, also can be used to describe the pressure distribution during steady flow of brine or water through an oil trap (or a fresh water aquifer containing an LNAPL) in which the aquifer material varies in texture along the path of flow. The relationships illustrated in Figure 4.8 hold for this case also. Therefore, it can be predicted that oil or LNAPL concentrations are highest in regions of coarse sands just upslope from finer-textured materials. The highest wetting fluid saturations are in the finer materials immediately upslope from coarser materials.

4.3. STEADY FLOW OF A NONWETTING FLUID

Steady flow of air is sometimes induced in the field to aerate polluted aquifers allowing biodegradation of a residual NAPL. In this case air is injected under pressure in a well penetrating an aquifer. The process is called "sparging." A flow of air may also be induced in the reverse direction, i.e., towards the well, in an attempt to remove contaminating chemicals as vapor mixed with the air. The latter process is called "soil vapor extraction."

Two Laboratory experiments are discussed in Chapter 3 involving steady flow of a nonwetting fluid. One involves the simultaneous flow of two fluid phases under the same pressure gradient imposed to obtain uniform saturations for the measurement of k_w and k_{nw} as functions of saturation. The other involves steady upward flow of air only, under a pressure gradient equal to the static pressure gradient in the liquid wetting phase. This is done in order to obtain k_{nw} on porous rock cores.

Another experiment, considered here, is an exercise to illustrate the physics of nonwetting phase flow emerging into the atmosphere from a partially saturated porous medium. This involves horizontal flow of a nonwetting fluid in the presence of a static wetting phase. A cylindrical core of porous rock of small diameter is considered. The surface of the cylinder is sealed except for the ends. The porous cylinder is first fully saturated with a liquid. A pressure difference is imposed in the air across the ends of the core sufficient to displace some of the liquid and to establish a flow of air through the medium. The air is pre-saturated with the vapor of the liquid, and the flow is initially assumed to reach a steady state so that the liquid retained in the porous rock eventually becomes static. The situation is illustrated in Figure 4.9.

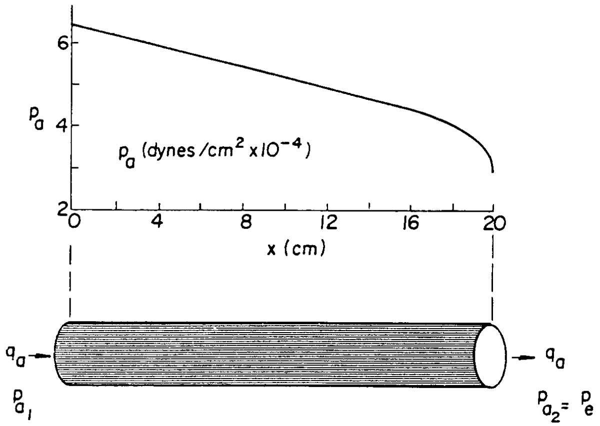

Figure 4.9. Horizontal porous cylinder containing static liquid and flowing air.

The version of Equation 4.3 applicable after the liquid becomes static is

$$\frac{dp_c}{dx} = -\frac{q_a \mu_a}{k_a}$$

where the subscript "a" refers to air. In order for air to pass entirely through the core, p_a must exceed p_ℓ by an amount at least equal to p_e for the particular medium and fluids considered. The subscript "ℓ" refers to liquid.

It is assumed that p_c at the outflow face is equal to p_e. Since the liquid is static, p_ℓ is a constant along the line of flow. It is convenient to use p_ℓ as a datum from which to measure the pressure of air because p_c is then equal to p_a, and k_a is a function of p_a. The air pressure at any point x along the centerline of the cylinder is given by

$$\int_{p_{a1}}^{p_a} k_a \, dp_a = -q_a \mu_a x \ .$$

The integral on the left side of this equation can be evaluated if $k_a(p_c)$ is known. One way of estimating this is to employ Equation 3.56. In any case, the integration can be performed numerically.

The pressure distribution shown in Figure 4.9 was determined by numerical integration using Equation 3.56 for $k_a(p_a)$ with $\lambda = 2$. Figure 4.9

indicates that much of the pressure drop occurs near the outflow face. This is because the liquid retained in the core tends to accumulate at the outflow face and resistance to flow at this face is large relative to resistance through other parts of the core.

The analysis described above is based on a steady state assumption that contradicts the conclusion. Air pressure must exceed the ambient pressure by an amount equal to the entry pressure before emerging from the medium, but at the instant air emerges its pressure is reduced to the ambient pressure. The only way this can occur with horizontal flow is for air to emerge in spurts, as if bubbles of air emerge from the medium. One must conclude that flow of air emerging from a partially saturated horizontal column cannot be steady, contradicting the initial assumption of a steady state. In fact, the latter conclusion is supported by visual observation.

The conclusion that liquid tends to concentrate at the outflow face where air exits the porous medium is also supported by observation. The tendency for water to accumulate at the outflow face is increased in the case of soil vapor extraction, because of the convergence of flow around the well bore. This presents a problem in conducting a vapor extraction operation, because the high resistance to air flow near the well bore reduces the rate at which vapor can be removed.

The tendency for air to preferentially flow through channels of the lowest entry pressure and highest permeability also reduces the efficiency of both sparging and vapor extraction. Pockets of wetting fluid and entrapped nonwetting fluid are likely to be by-passed by the channels of flowing air. Vapor extraction and biodegradation are both limited by the rates at which constituents can be interchanged by diffusion between the bypassed pockets and the air channels.

4.4. STEADY FLOW TOWARD PARALLEL DRAINS

The problem considered here is to a large extent hypothetical although it has considerable practical importance. In many parts of the world, agricultural drains are designed (with respect to depth and spacing) by assuming water percolation to the water table is steady. Upon reaching the water table, water flows laterally to some sink. In order to maintain the water table at a safe depth below the surface, artificial sinks may be provided in the form of parallel drains at some depth below the surface, preferably near a low-permeable stratum if such a stratum exists.

The hypothetical model usually used as a basis of design is illustrated in Figure 4.10. This model is idealized in several respects. First, the recharge rate w is assumed to be constant in respect to both time and areal distribution. Secondly, the drains are considered to act as open ditches penetrating to some constant depth, whereas in reality, the drains are usually buried conduits of relatively small size, say 15 to 20-cm tile. The water table is regarded as being in equilibrium under the existing conditions of recharge and drainage. It also is assumed that the Dupuit approximations are valid for this case and that the aquifer is homogeneous and isotropic.

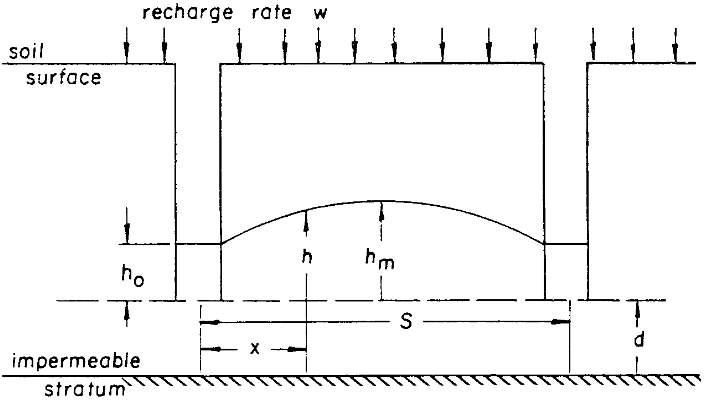

Figure 4.10. Flow toward parallel drains with uniform recharge.

For the usual analysis of this problem, it is assumed that the water table is the upper streamline. This latter assumption may often lead to serious error, and the purpose here is to show how this assumption can be avoided. However, the standard analysis is presented first.

The origin of coordinates is placed on a layer, assumed to be impermeable, below the center of the drain on the left. A vertical plane drawn between the center of the two drains at x = S/2 is regarded as a divide across which no flow takes place. To the left of the central divide, all flow eventually reaches the left drain, and to the right of the divide flow is toward the right drain.

Flow through a vertical plane at x is considered. The flow through this section is toward the left drain and is equal to the recharge on the surface between the section and the divide. The surface area on which this recharge takes place (for a slice of aquifer of unit thickness) is equal to S/2 - x. The quantity of water flowing per unit time through the vertical plane of unit width is, therefore,

$$\mathbf{Q_x = w\left(\frac{S}{2} - x\right)}\ .$$

According to the Dupuit approximation, Q_x should also be given by

$$Q_x = Kh\frac{dh}{dx} ,$$

where h is the elevation of the water table above the impermeable layer. Equating the two expressions for Q_x and integrating with respect to x gives

$$\left(\frac{wS}{2}\right) x - \left(\frac{w}{2}\right) x^2 = \frac{K}{2}\left(h^2 - h_o^2\right) . \qquad 4.12$$

In the usual case $h_o \ll h_m$ so that h_o can be considered as equal to zero. In this case, the flow equation can be integrated between the limits (x = 0, h = d) and (x = S/2, h = h_m + d) which gives

$$S^2 = \frac{4Kh_m}{w}\left(h_m + 2d\right) . \qquad 4.13$$

Equation 4.13 has been used in many parts of the world and especially by the U.S. Soil Conservation Service for determining the spacing of relief drains. The equation is sometimes known as the ellipse equation. In other places it is called the Hooghoudt, Donnan, or steady state equation. It is more appropriate for areas of frequent rainfall, such as Holland where it originated, than for irrigated regions. In such cases, the assumption of a constant recharge and an equilibrium state of the water table may be a reasonable approximation.

If the height d is not relatively small compared to h_m, the Dupuit approximation fails rather seriously because flow into the drain approaches radial rather than horizontal. In the latter case, a different equation for the drain spacing is advised. For example, according to Equation 4.13, as d becomes infinite, S also becomes infinite. This is not realistic, the difficulty being that the equation fails to account properly for radial flow in the vicinity of the drain.

Equation 4.13 also fails for cases in which the so-called capillary fringe $p_e/\rho_w g$, designated by h_e is not small compared to h_m. This is a frequent situation, but for many years little attention was given to accounting for the flow in the region above the water table. The following analysis was suggested by Duke (1973) as a way of accounting for this flow.

The capacity of the region above the water table to conduct water horizontally toward a drain is given by an expression suggested by Myers and van Bavel [Bouwer (1964)] as

$$H_k = \frac{1}{K_m}\int_0^H K_w \, dh \qquad 4.14$$

where H_k is the "effective permeable height," a hypothetical height of saturated soil having the same capacity to transmit water as does the partially saturated region above the water table. The maximum field conductivity is designated by K_m and K_w is the conductivity at any other saturation. The elevation above the water table is designated as h, and H is the elevation to the soil surface.

When $H < h_e$, the soil remains saturated to the surface and $H_k = H$. If $H > h_e$, K_w is dependent on h and any vertical flux that may exist. During steady percolation to a water table, h_c is less than h at every point. Figure 4.3 illustrates this situation.

The integration indicated in Equation 4.14 can best be performed in two steps:

$$\mathbf{H_k = \frac{1}{K_m}\int_0^{h_e} K_w\, dh + \frac{1}{K_m}\int_{h_e}^{h_s} K_w\, dh} \;. \qquad 4.15$$

The limit h_s refers to the value of h_c at the soil surface. In the usual case, the surface is dry and $h_s \rightarrow \infty$, the value of the integrand at this limit being equal to zero.

In order to evaluate the integrals, however, the variable of integration h must be expressed in terms of h_c. This is accomplished by application of Equation 4.3 in the form

$$\mathbf{\frac{dh_c}{dh} = 1 - \frac{q_w}{K_w}}$$

so that

$$\mathbf{dh = \frac{dh_c}{1 - \frac{q_w}{K_w}}} \;.$$

For the first term on the right in Equation 4.15, K_w is K_m, a constant. Therefore, this term is

$$\mathbf{\frac{h_e}{1 - \hat{q}}}$$

in which $\hat{q}$ is the scaled flux q_w/K_m. For the second term

$$\mathbf{K_w = K_m\left(\frac{h_e}{h_c}\right)^{\eta}}$$

so that

$$\mathbf{H_k = \frac{h_e}{1-\hat{q}} + \int_{h_e}^{h_s} \frac{dh_c}{\left(\frac{h_c}{h_e}\right)^{\eta}\left[1-\hat{q}\left(\frac{h_c}{h_e}\right)^{\eta}\right]}} \; . \qquad 4.16$$

Equation 4.16 can be solved easily by numerical integration. It is sufficient to assume that H_k is a constant at all values of x (see Figure 4.10). The reason is that the integrand in Equation 4.16 becomes very small when h_c is substantially larger than h_e , say by a factor of about 2 or 3. Consequently, the greater depth to the water table near the drains has negligible effect on the value of H_k.

For a case of steady flow to parallel drains, with a steady percolation rate, the variable h in the ellipse equation (in particular, h_m) can be replaced with h + H_k. This procedure was used by Duke to calculate the water table profile in a laboratory drainage model consisting of a soil flume. The results are shown in Figure 4.11. In this figure, the measured water table elevations are shown along with those calculated by numerical solution of Equation 4.12 with h replaced by h + H_k. Evidently this procedure provides a better representation of the measured data than is obtained from Equation 4.12 in its original form.

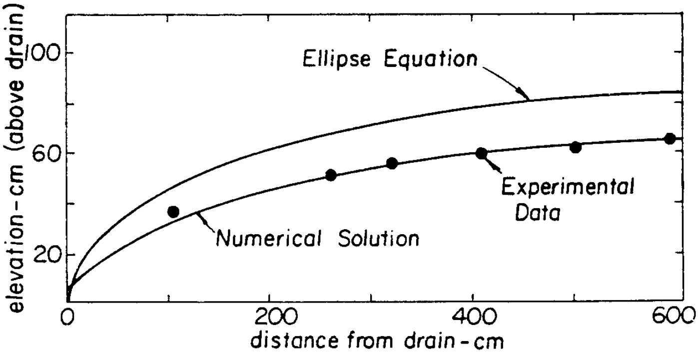

Figure 4.11. Numerical solution of Equation 4.12 corrected with Equation 4.16. [Adapted from Duke (1973)].

PROBLEMS AND STUDY QUESTIONS

l. Equation 4.4 does not imply a summation over 3 orthogonal directions. However, Equation 4.1 necessarily implies a summation over 3 orthogonal directions. Explain why this is necessary, physically.

2. In the process of geophysical exploration for oil, the geophysicists look for structural "highs;" that is, they check to see if certain rock strata are encountered at higher elevations than the known elevation of the corresponding strata in surrounding areas. Explain.

3. In a confined aquifer containing both oil and brine, in which the aquifer texture is at places relatively fine-grained and in other places coarse-grained, wells that are drilled into the fine-grained material may produce mostly brine. Explain.

4. Consider a vertical column of sand initially fully saturated with water and with a water table maintained at the base of the column. Fuel oil (specific gravity = 0.75) was then spilled at the top of the column until most of the water was removed from the column. Subsequently a water flood from the top of the column was initiated to remove as much of the fuel oil as possible. Calculate the piezometric gradient required to insure all mobile oil is flushed downward and out of the column.

5. In reference to the column in Problem 4, suppose one removed only part of the mobile oil by establishing a steady downward flow of water at a smaller rate than the critical rate.

    (1) Describe the oil distribution in the column when steady state is achieved.

    (2) Explain qualitatively what happens to the oil distribution after the flushing operation is terminated.

    (3) What additional information is needed about the column, the oil, and the sand to determine the distribution of oil when equilibrium is reestablished after the downward flow of water is terminated?

(4) How would your answers to questions (2) and (3) change if the water table at the base of the column, as well as the water flood from the surface, is terminated and both ends of the column are exposed to the atmosphere but protected from evaporation?

6. Consider an unconfined aquifer consisting of fully saturated sand with an entry pressure head (DNAPL into water) of 5 cm of water. There is a horizontal layer of finer textured sand at some depth below with $h_e = 15$ cm of water. When a sufficient amount of DNAPL (S.G. = 1.2) is spilled at the surface of the aquifer, a mound of DNAPL builds up over the fine-textured layer.

   (1) Calculate the maximum height the mound could reach before DNAPL enters the fine-textured layer.

   (2) Describe qualitatively what happens to the mound after the spilling stops. Will the DNAPL spread for an indefinite distance?

   (3) When the spreading stops, what will be the maximum thickness of the mound?

   (4) Will all of the DNAPL be in the zone of interconnected DNAPL at the time spreading stops? Explain.

7. Consider a soil profile in which there is a junction between a coarse and a finer-textured material. Assume that both the coarse and the finer material are partially desaturated. Compare

   (1) the capillary pressures,

   (2) the saturations, and

   (3) the values of K_w of the two materials at the junction. Explain.

8. Given that a sand has an entry pressure head of about 30 cm, estimate the length of column necessary to obtain $k_w(p_c)$ by the long-column method. How would the necessary length of column be affected if the soil under investigation is highly structured (with a low value of η being expected)? Explain.

9. When using a long column to determine $k_w(p_c)$ there is a danger that upon decreasing the flow rate by some increment, the column may

over-desaturate at the top and later resaturate to some degree. Explain. This tendency is greater when columns are not supplied with sufficient air vents. Explain.

10. The tendency indicated in question number 9 may not affect the results significantly if the method is used to obtain $k_w(S)$. Explain.

11. The length of column needed to obtain $k_w(p_c)$ by the short-column method is affected by η, but not by the entry pressure head. Explain.

12. Consider a case of steady downward flow of water in homogeneous sand in a long column to a water table below. If the value of q_w is 10^{-8}cm/s and K_m is 10^{-4} cm/s, what is K_{rw} in the upper portion of the column?

13. Consider a case of water moving steadily downward through a very long column of homogeneous sand ($\lambda = 2$ and $S_r = 0.2$) which is partially saturated. The upper half of the column has a diameter of 3 cm and the lower half has a diameter of 6 cm. The value of q for the upper portion is 1.2×10^{-3} cm/s and $S_e = 0.8$. Ignore the situation existing near the junction of the upper and lower half and estimate the following:

    (1) the value of k_w of the soil in the lower half in square microns,

    (2) the value of S for the lower half,

    (3) the value of k_{rw} for the upper and the lower portion,

    (4) the ratio of the p_c in the upper half to that in the lower half.

14. When using a soil water extractor to determine the rate at which leachate passes a horizontal plane in a soil profile, it is important that streamlines do not converge or diverge toward the extracting device. If the extractor consists of an interceptor operating at atmospheric pressure, would you expect the streamlines to diverge or converge toward the device under usual soil water conditions? Explain.

15. Evapotranspiration rates may fluctuate greatly from hour to hour depending upon atmospheric conditions, whereas evaporation from a nearby fallow soil may not fluctuate at all during the same period of time. Explain.

16. Explain why layered soils typically drain much more slowly than homogeneous soils. Describe conditions under which a coarse stratum may control the percolation rate, and other conditions in which a fine-textured stratum may be the major restriction.

17. The vertical distribution of soil water obtained with a trickle irrigation system is likely to be much more favorable than that obtained by a sub-irrigation system. Explain fully.

18. Would you expect the effective permeable height to be greater or less if the recharge rate w is increased, other conditions being equal?

19. Consider a horizontal cylinder (of small diameter) of homogeneous porous rock. The medium is initially saturated with water and is open to the atmosphere. Water is maintained at atmospheric pressure at the inflow end of the cylinder. The water pressure at the outflow end is maintained at a very low value (very high capillary pressure) and the outflow approaches a maximum q_m.

    Assuming that $p_c < p_d$, $K_w = K_m$, and for $p_c \geq p_d$,

    $$\mathbf{K_w = K_m (p_d / p_c)^{\eta}},$$

    show that

    $$\mathbf{q_m = K_m \frac{\eta}{\eta - 1} \frac{p_d}{\rho g L}}$$

    in which L is the length of the cylinder of porous rock.

20. In reference to problem number 19, assume the rock has a very uniform pore size and $p_d / \rho g$ is twice the length of the porous cylinder. Estimate k_m in square centimeters if q_m is 10^{-4} cm/s at a room temperature of about 20 degrees centigrade or slightly less. What would q_m be if T = - 10 degrees C?

Chapter 5

UNSTEADY FLOW OF IMMISCIBLE FLUIDS

5.1. UNSTEADY FLOW WITH TWO FLUIDS

Variables pertinent to a description of unsteady flow in porous media containing two fluids include pressure, flux rate and permeability of each fluid along with saturation, space coordinates and time. Fluid properties including density and viscosity are treated as constants, although a more general analysis would treat them as variables. Medium properties including porosity and maximum permeability also are treated as constants. The effective permeabilities are functions of capillary pressure but are considered invariant in respect to direction since the medium is assumed to be isotropic.

For isotropic media, the generalized flux Equation 3.44 is written for each fluid as

$$q_{wi} = \frac{k_w}{\mu_w}\left[-\frac{\partial p_w}{\partial x_i} + \rho_w g_i\right] \qquad 5.1$$

and

$$q_{nwi} = \frac{k_{nw}}{\mu_{nw}}\left[-\frac{\partial p_{nw}}{\partial x_i} + \rho_{nw} g_i\right]. \qquad 5.2$$

For fluids undergoing negligible changes in density, material balance equations are obtained by considering a reference volume of the bulk medium, including the solid matrix as well as a representative portion of each of the two fluids. Such a volume is of a size sufficient for the specification of porosity as explained in Section 1.3.1. The resulting material balance equations are

$$\phi\frac{\partial S}{\partial t} = -\frac{\partial q_{wi}}{\partial x_i} = \frac{\partial q_{nwi}}{\partial x_i}. \qquad 5.3$$

From the definition of capillary pressure,

$$\frac{\partial p_{nw}}{\partial x_i} - \frac{\partial p_w}{\partial x_i} = \frac{\partial p_c}{\partial x_i} \,. \qquad 5.4$$

Problems involving displacements of one fluid by another are analyzed by solving Equations 5.1-5.4 simultaneously, with appropriate initial and boundary conditions. Examples include imbibition of water in soils, infiltration, drainage of soils and the displacement of one liquid by another, e.g., oil by brine or an LNAPL by water. Solutions require the relationships, $k_w(p_c)$, $k_{nw}(p_c)$, and $S(p_c)$ to be known or estimated, and usually a solution requires numerical integrations.

Many practical problems can be analyzed by solving simpler equations because a number of the factors accounted for in the complete two-phase equation set are not significant in all cases. A simplification often made by soil scientists for application to water-air displacements in soils is to neglect the density and pressure of air.

An assumption of constant air pressure is applicable where the resistance to air flow is negligible in all parts of the flow system, and consequently, capillary pressure is essentially equal to the negative of the water pressure. Since the viscosity of water is more than 50 times the viscosity of air this assumption is often justified. However, there are regions in many air-water systems where k_w is much greater than k_{nw} in which case the assumption is not justified. Obviously it is not justified, for example, in the case discussed in Section 4.3, because air emerging from the porous sample must exit through a region of high liquid saturation.

A contrasting simplification is often made by petroleum engineers when dealing with displacements of petroleum fluids, especially oil by brine. The two fluids flow side by side over large distances with very small gradients of capillary pressure and saturation. The driving force causing flow is large, and nearly the same for both fluids. The two fluids often have comparable viscosities at the temperature existing in the aquifer. Furthermore, the slopes and the density difference are often small so that buoyancy, $\Delta\rho g_i$, may be negligible.

Under the latter conditions, the two fluids are assumed to flow together under the same pressure gradient. In most oil-brine aquifers, however, there are regions where the saturation changes substantially over relatively short distances. An analysis of saturation changes in such regions requires the retention of both pressure gradients in Equations 5.1 and 5.2.

A statement frequently made by petroleum scientists is that capillary, as well as buoyancy effects are disregarded. In fact, capillary effects are not entirely ignored. It is only the contribution of capillary pressure gradients and buoyancy to the driving force causing flow that is ignored. The effect of capillarity on k_w and k_{nw} remains implicit in their equations.

5.2. FLOW OF WATER AND AIR IN SOILS

Assuming that the density and viscosity of air are negligible, and that wherever an interconnected air phase exists, the pressure of air is everywhere atmospheric. Equations 5.1-5.4 can be combined to give

$$\phi \frac{\partial S}{\partial t} = -\frac{\partial}{\partial x_i}\left[\frac{k_w}{\mu_w}\left(\frac{\partial p_c}{\partial x_i} + \rho_w g_i\right)\right]$$

which is usually written as

$$\frac{\partial \theta}{\partial t} = -\frac{\partial}{\partial x_i}\left[K\left(\frac{\partial \psi}{\partial x_i} - \frac{\partial h}{\partial x_i}\right)\right], \qquad 5.5$$

where $\psi \equiv - p_w / \rho_\omega g$ and h is the elevation above a datum. Equation 5.5 is known as the Richards equation after L. A. Richards (1931). The Richards equation is further simplified for special cases.

5.2.1. Horizontal Linear Imbibition without Air Resistance

The case of horizontal linear imbibition of water into a homogeneous soil is of considerable academic interest and has been studied often. The additional simplifications made in this case include neglecting the gravity force and assuming that θ is a single-valued function of ψ only. The latter assumption eliminates the possibility of analyzing flow into soils which are non-homogeneous or soils where hysteresis may occur.

The particular case most often analyzed is that of imbibition of water from a source into a soil (with uniform water content θ_i), while maintaining a constant water content θ_o at the plane where $x = 0$. It is required that θ_o satisfy the condition

$$\theta_m > \theta_o > \theta_i,$$

where θ_m is the maximum water content for the soil. If the conditions described above are satisfied, Equation 5.5 can be simplified as

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial\theta}{\partial x}\right) \qquad 5.6$$

where

$$D \equiv -K\frac{d\psi}{d\theta} ,$$

and is called the diffusivity coefficient. Defining D in terms of the total derivative of ψ is based on the assumption that ψ is a single-valued function of θ only.

Equation 5.6 is nonlinear, because D is a function of θ. A solution can be found by first reducing the equation to an ordinary form with a suitable transformation of variables, i.e., the Boltzman transformation given by

$$z(\theta) = xt^{-1/2} .$$

With this transformation, Equation 5.6 becomes

$$-\frac{z}{2}\frac{d\theta}{dz} = \frac{d}{dz}\left(D\frac{d\theta}{dz}\right) . \qquad 5.7$$

The initial and boundary conditions considered are:

$$\theta(0,t) = \theta_o ,$$

$$\theta(\infty,t) = \theta_i ,$$

$$\theta(x,o) = \theta_i .$$

Under the transformation, these become

$$\theta(z = 0) = \theta_o ,$$

$$\theta(z \rightarrow \infty) = \theta_i .$$

Since the boundary and initial conditions (as well as the starting equation) are reduced by the transformation,

$$x(\theta) = z(\theta)t^{1/2} \qquad 5.8$$

is a valid solution of Equation 5.6 for the conditions specified. This implies that the distance to a plane where θ has a specified value is proportional to the square root of time, provided that the conditions are satisfied. In this case $z(\theta)$ is a single-valued function of θ satisfying Equation 5.7. Equation 5.8 is known as the square root of time law.

Equation 5.8 also implies that the wetted region has a saturation profile that stretches with time but remains similar in shape. Consequently, it seems reasonable to suppose that the cumulative inflow should also be proportional to $t^{1/2}$. In fact, this can be shown to be true based on theory which has been verified, at least roughly, by many investigators. Figure 5.1 shows some typical results obtained by King (1964) for conditions as near as possible to those assumed for the derivation of Equation 5.8.

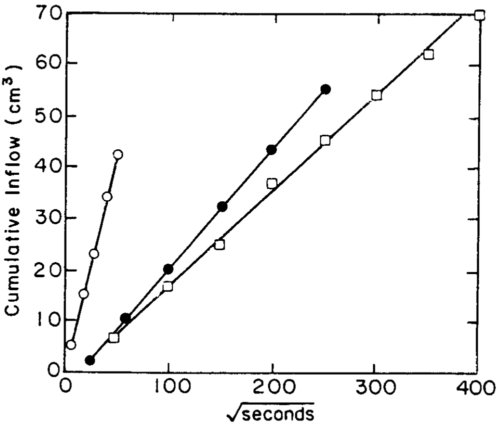

Figure 5.1. Cumulative volume of inflow as a function of time during horizontal linear imbibition [King (1964)].

Because of the very restrictive conditions for which Equation 5.8 is derived, experimental verification is difficult. According to King (1964), the difficulties include:

(1) The soil must be packed in a homogeneous manner.

(2) The vertical dimension of the tube must be sufficiently small that the effect of gravity on the distribution of water is negligible. King stated that the vertical dimensions must be small compared to $p_d/\rho_w g$. In other words, the tube for a coarse-textured soil must be smaller than for a fine-textured soil.

(3) The value of ψ at the inflow boundary must be constant and larger than $p_e / \rho_w g$.

The latter condition is particularly difficult to satisfy rigorously because water must be admitted through a capillary barrier. The resistance to flow through such a barrier and through a contact region must be compensated for in some way. However, the resistance of the barrier and the contact region vary with the flow rate, and also with time because of plugging. Failure to compensate for the resistance may result in curves, such as those shown in Figure 5.1, having a small intercept on the time axis. In fact, it is practically impossible to satisfy the theoretical conditions rigorously, because Equation 5.8 implies an infinite flux rate at t = 0. This cannot be achieved physically, especially through a capillary barrier.

In the derivation of Equation 5.8 described above it is assumed that θ is a single-valued function of ψ and that the resistance to air flow is negligible. In fact, an equation of the same form as Equation 5.8 can be derived without the single-value assumption. To demonstrate that θ need not be a single-valued function of ψ to obtain Equation 5.8, consider the imbibition of water into a horizontal tube as shown in Figure 5.2.

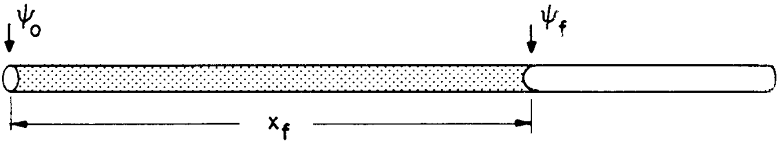

Figure 5.2. Imbibition into a horizontal capillary tube.

For the capillary tube in Figure 5.2, the flux rate q is given by

$$q = -K^* \left[\frac{\psi_f - \psi_o}{x_f} \right] = \frac{c}{x_f},$$

where c is a constant, the subscript f refers to the wetting front, and

$$K^* = \frac{r^2 \rho_w g}{8 \mu_w}$$

in which r is the radius of the tube. A continuity equation referring to the entire wetted volume of the tube is given by

$$q = c_2 \frac{dx_f}{dt}.$$

Consequently,

$$\frac{dx_f}{dt} = \frac{c}{x_f}.$$

Integration results in

$$x_f^2 = c't,$$

or

$$x_f = c''t^{1/2}. \qquad 5.9$$

Equation 5.9 implies that both the distance to the front and the volume of water imbibed increase in proportion to $t^{1/2}$. In this case, the value of θ behind the wetting front is constant although ψ varies from θ_o to ψ_f. Therefore, θ is not a single-valued function of ψ. An analogous result can be obtained assuming a homogeneous tube of soil having a uniform pore size. It would seem, therefore, that the validity of the square root of time law does not imply that θ is necessarily a single-valued function of ψ.

In order for Equation 5.6 to be of any use beyond the derivation of Equation 5.8, it must be possible to measure the function $D(\theta)$. R. R. Bruce and A. Klute (1956) have presented a procedure for making such a measurement. The measurement is made by performing experiments which satisfy the conditions for which Equations 5.7 and 5.8 are valid.

Equation 5.7 is integrated between limits with respect to $z(\theta)$. The result is

$$-\int_{\theta_i}^{\theta_x} \frac{z}{2} d\theta = D(\theta_x)\left(\frac{d\theta}{dz}\right)_{\theta_x} - D(\theta_i)\left(\frac{d\theta}{dz}\right)_{\theta_i}. \qquad 5.10$$

The last term on the right of Equation 5.10 is zero at constant time because the derivative $(d\theta/dz)_{\theta_i}$ is zero at values of x large enough to be ahead of the wetting front.

It follows that

$$D(\theta_x) = -\frac{1}{2}\left(\frac{dz}{d\theta}\right)_{\theta_x} \int_{\theta_i}^{\theta_x} z\,d\theta\,.$$

Replacing z with its equivalent in terms of x and t, and considering a constant t results in

$$D(\theta_x) = -\frac{1}{2t}\left(\frac{dx}{d\theta}\right)_{\theta_x} \int_{\theta_i}^{\theta_x} x\,d\theta. \qquad 5.11$$

Equation 5.11 provides the basis for the following procedure for evaluating $D(\theta)$:

(1) Obtain $\theta(x)$ at a particular t.

(2) Evaluate the integral in Equation 5.11 graphically or numerically.

(3) Calculate D at values of θ_x, thereby obtaining $D(\theta)$.

The reader is referred to the paper by Bruce and Klute (1956) for details of the experimental procedure.

5.2.2. Infiltration from a Constant Head Source

The term "infiltration" designates vertical flow of water into a soil from a source at the surface. Resistance to air flow in response to the advancing water may or may not be important. Infiltration is a common physical phenomenon and is a major process of the hydrologic cycle.

Assuming that resistance to air displacement can be neglected, the downward flow can be described by Equation 5.5. For this case, Equation 5.5 becomes

$$\frac{\partial\theta}{\partial t} = -\frac{\partial}{\partial z}\left[K\left(\frac{\partial\psi}{\partial z}+1\right)\right] \qquad 5.12$$

where z is the vertical coordinate measured positive downward, i.e., positive in the direction of flow.

The term in parenthesis in Equation 5.12 represents the combined driving force (per unit weight of water) due to the pressure gradient and gravity. The vertical component of the pressure gradient is a result of capillary effects due to the variation of θ with z. The gravity force per unit weight of water in a vertically downward direction is 1.0. As an aid in visualizing the way the two forces are interrelated as water infiltrates downward through the soil, an idealized model consisting of a vertical capillary tube is considered first. The idealized model is shown in Figure 5.3.

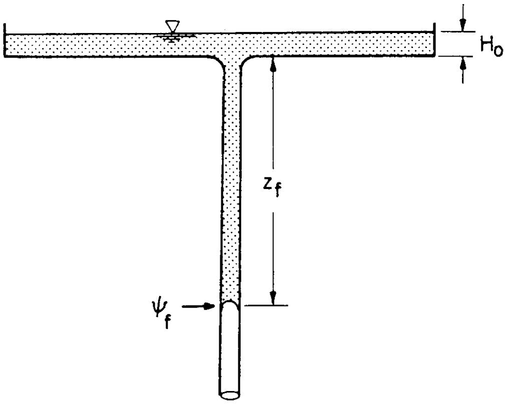

Figure 5.3 Downward imbibition into a capillary tube from a constant head source.

For the capillary tube shown in Figure 5.3, the flux rate I is given by

$$\mathbf{I} = \mathbf{K}^* \left[\frac{\mathbf{H_o + z_z + \psi_f}}{\mathbf{z_f}} \right] \qquad 5.13$$

in which K^* is a constant depending on the fluid properties and the radius of the tube. H_o is the depth of ponded water that is maintained constant, z_f is the distance to the air-water interface from the source and ψ_f is the value of $-p_w/\rho_w g$ at the interface.

For simplicity, a case is considered for which H_o is small enough to be neglected so that

$$\mathbf{I} = \mathbf{K}^* \left[\frac{\psi_f}{z_f} + 1 \right] \quad ,$$

where ψ_f/z_f represents the capillary force per unit weight of water, and 1.0 is the gravity force per unit weight of water. When z_f is small, the capillary force is relatively large and may be more important than gravity. However, the capillary force decreases with z_f and eventually becomes negligible, whereas the gravity force remains constant. Consequently, at large times, the flux rate approaches the constant K^*.

An equation of the same form as Equation 5.13, for infiltration into soils from a ponded water source, was derived by Green and Ampt (1911). For this case, K^* is considered to be K_m, the maximum conductivity existing after the soil is flooded, and ψ_f has been given various interpretations. Whisler and Bouwer (1970) recommended the use of $p_e/\rho_w g$ for ψ_f, and suggested a way of measuring p_e directly in the field. This is an obvious choice since z_f should correspond to an elevation where p_c is p_e.

Equation 5.13 provides a relationship between I and z_f but does not provide any information about the time at which a particular I will occur or about the distribution of θ with respect to z and t. To obtain this information, it is necessary to introduce a continuity equation. Green and Ampt assumed that all the water infiltrated is held by the soil above the elevation where ψ is ψ_f and p_c is p_e. This is a simplification of the real situation since some water advances ahead of the front. Consequently, the Green and Ampt equation in its original form, can give no information about the distribution of θ in the partially saturated region.

The continuity equation used by Green and Ampt is

$$\mathbf{I} = \left(\theta_m - \theta_i\right) \frac{dz_f}{dt} \qquad 5.14$$

where θ_m and θ_i are the final and initial water contents respectively. Combining Equations 5.13 and 5.14 and integrating with respect to t gives

$$t \approx \frac{V}{K_m} - \frac{H_t \Delta\theta}{K_m} \ln\left(1 + \frac{V}{H_t\,\Delta\theta}\right) \qquad 5.15$$

in which H_t is the sum $H_o + \psi_f$, and V is the volume/area of water infiltrated in time t, i.e., $z_f\,\Delta\theta$.

The use of Equation 5.15 requires a knowledge of:

(1) the initial water content θ_i,

(2) maximum water content θ_m,

(3) hydraulic conductivity K_m,

(4) the sum $H_o + \psi_f$.

Whisler and Bouwer (1970) have suggested methods of measuring the required quantities in the field. Whisler and Bouwer (1970) and later Mein and Larson (1973) have presented results indicating that the Green and Ampt equation agrees well with measured data. Considering the assumptions made in its derivation, this equation would be expected to have greater validity for soils with a relatively narrow range of pore sizes. In such soils, the quantity of water advancing beyond the front where ψ is ψ_f should be minimal, especially at early times, and the resistance to air flow ahead of the front also should be minimal.

Philip (1957) was among the first to solve Equation 5.12 for the case of infiltration from a constant head source. He first wrote this equation in the θ-dependent form

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial \theta}{\partial z}\right) - \frac{\partial K}{\partial z}, \qquad 5.16$$

where z is the vertical coordinate, positive downward. For the case considered, the conditions are:

$$t = 0,\ z > 0,\ \theta = \theta_i$$

$$t \geq 0,\ z = 0,\ \theta = \theta_o$$

in which θ_i is the initial water content and θ_o is a water content maintained at the surface by a constant ψ at $z = 0$.

Equation 5.16 is converted into a z-dependent form by multiplying each term by $\partial z / \partial \theta$, and using the identity

$$\left(\partial\theta/\partial t\right)_z \bullet \left(\partial z/\partial\theta\right)_t \equiv -\left(\partial z/\partial t\right)_\theta$$

so that

$$-\frac{\partial z}{\partial t} = \frac{\partial}{\partial \theta}\left(D\frac{\partial \theta}{\partial z}\right) - \frac{\partial K}{\partial \theta}, \qquad 5.17$$

which is subject to the conditions pertaining to Equation 5.16.

The solution of Equation 5.17 is considered to be a perturbation of the solution of

$$-\frac{\partial z^*}{\partial t} = \frac{\partial}{\partial \theta}\left(D\frac{\partial \theta}{\partial z^*}\right), \qquad 5.18$$

where z^* is taken as a first estimate of z. The solution of Equation 5.18 is analogous to Equation 5.11, i.e.,

$$z^* = \alpha(\theta)t^{1/2} . \qquad 5.19$$

Subtracting Equation 5.17 from Equation 5.18 gives

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial \theta}\left(D\frac{\partial \theta}{\partial z^*}\frac{\partial y}{\partial z}\right) + \frac{\partial K}{\partial \theta} \qquad 5.20$$

where

$$y \equiv z - z^* .$$

The approximation

$$\frac{\partial y}{\partial z} \approx \frac{\partial y}{\partial z^*}$$

leads to

$$\frac{\partial y^*}{\partial t} \approx \frac{\partial}{\partial \theta}\left[D\left(\frac{\partial \theta}{\partial z^*}\right)^2\frac{\partial y^*}{\partial \theta}\right] + \frac{\partial K}{\partial \theta}, \qquad 5.21$$

where y^* is used in place of y because of the approximation. The transformation

$$\chi \equiv y^* t^{-1}$$

leads to

$$\chi = \frac{d}{d\theta}\left(P\frac{d\chi}{d\theta}\right) + \frac{dK}{d\theta} \qquad 5.22$$

in which

$$P(\theta) \equiv D\left(\frac{d\theta}{d\chi}\right)^2 .$$

From Equation 5.19

$$\alpha(\theta) = z^* t^{-1/2}. \qquad 5.23$$

Integrating Equation 5.22 with respect to θ gives

$$\int_{\theta_i}^{\theta} \chi\, d\theta = P\frac{d\chi}{d\theta} + (K - K_i), \qquad 5.24$$

where K_i corresponds to θ_i. The conditions are

$$\theta = \theta_i \text{ for } \chi = 0.$$

Equation 5.24 is solved numerically for this condition. A new residual error

$$y^{**} \equiv y - y^*$$

is introduced, and the procedure is repeated with new residuals as often as required to give the necessary accuracy.

The solution finally is developed as a power series in $t^{1/2}$:

$$z(\theta) = \alpha t^{1/2} + \chi t + \psi t^{-3/2} + \ldots + f_m(\theta) t^{m/2} + \ldots \qquad 5.25$$

where the coefficients of $t^{m/2}$ are functions of θ which are solutions of a series of ordinary equations that can be solved numerically. For details of the method of calculation, the reader is referred] to the paper by Philip (1957).

According to Philip (1957), within the range of variables most often encountered, only the first few terms of Equation 5.25 are needed since the series converges rapidly. For practical applications, the first two terms on the right are often considered sufficient. With this simplification, Equation 5.25 can be shown to be in approximate agreement with the Green and Ampt equation where $z(\theta) = z(\theta_m)$. In this case,

$$I \approx \Delta\theta \frac{dz(\theta_m)}{dt}.$$

At large times, $I \rightarrow \chi$ where χ is an approximation of K_m. At very early times, the first term is more important so that the infiltration tends to proceed as it would for the horizontal case, indicating that capillary forces are predominant at early times.

5.2.3. Infiltration with Air Compression

Infiltration occurring when the soil contains an underlying water table, or some other restrictive layer delaying the escape of air, results in compression of the air ahead of the wetted front. The effect of this is an increase in the air pressure resulting in a decrease in infiltration rate, frequently followed by an abrupt breakthrough of air to the surface. Youngs and Peck (1964), Peck (1965) and McWhorter (1971) are among those who have studied this situation experimentally and theoretically.

The observations of the latter investigators indicate that the threshold pressure at which air breaks through to the surface is significantly larger than that which is indicated from the primary loop of a wetting $\psi(\theta)$ curve. Evidently, a region of $\theta \approx \theta_m$ moves into the soil during compression, and at the time of breakthrough, there is a slight desaturation of this region so that the threshold pressure observed is one that occurs on a desaturation loop starting with θ_m.

Prior to breakthrough, there is a substantial but gradual reduction in infiltration rate. After breakthrough, a counterflow of air begins, accompanied by an immediate increase in the infiltration rate. At this time there is often a sharp decrease in air pressure which is thought to be caused by a disturbance of the soil at the time of air breakthrough. McWhorter (1971) tested this hypothesis by conducting an independent experiment with a column of consolidated Berea sandstone in which no decrease in air pressure was observed at the time of breakthrough.

Experimental results obtained by McWhorter (1971) are shown in Figures 5.4 and 5.5, illustrating the phenomena described above.

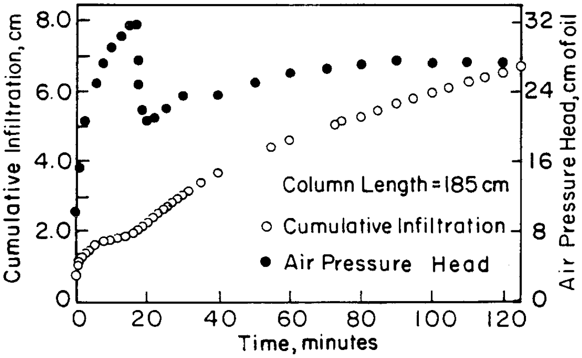

Figure 5.4. Infiltration into a column of Poudre sand (185-cm length) with air compression followed by air counterflow.

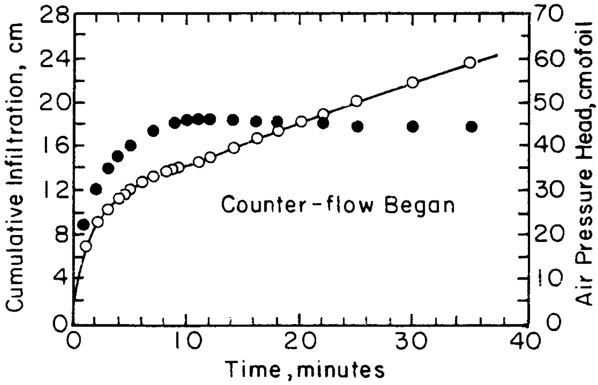

Figure 5.5. Infiltration into a column of Berea sandstone (17.4-cm length) with air compression followed by air counterflow.

McWhorter derived an equation that was successful in predicting the cumulative inflow into the columns. The reader is referred to the paper by McWhorter for a detailed discussion of this theory.

5.2.4. Constant Rate Infiltration

In the preceding sections dealing with infiltration, a condition of constant water pressure or constant water content is imposed at the soil surface. Imposing a constant value of q_w at the surface is of equal interest. However, the latter condition has not received as much theoretical study as the former case. Perhaps this is because the constant rate case is not amenable to the type of solutions that can be used for the constant head case. Several cases of constant rate are of interest:

(1) infiltration into a semi-infinite homogeneous profile with no restrictive layer, at a rate smaller than K_m so that no ponding occurs,

(2) infiltration into a similar profile at a rate exceeding K_m and producing ponding after a period of time,

(3) infiltration into a soil profile with a restriction at some depth, such as a water table, producing a counterflow of air toward the surface.

The first of the cases listed has been investigated by Rubin and Steinhardt (1963), using numerical methods for solving Equation 5.16 subject to the constant rate boundary condition. Parlange (1972) obtained an analytical solution of the same problem after first rewriting Equation 5.16 in the z-dependent form, i.e., Equation 5.17, as Philip (1957) had written it earlier for the constant water-content condition. Parlange obtained his solution by using a method of successive approximations described by Ames (1965), and employed by McWhorter (1971 and 1976) for solving 2-phase flow problems. The results of Parlange's analytical solution agree with the numerical results of Rubin and Steinhardt and confirm the qualitative observations noted by numerous experimental investigators.

For a case in which q_w at the surface is less than K_m and resistance to air flow is a negligible factor, the qualitative observations are as follow:

(1) After a period of time such that water has infiltrated to a substantial depth, a region of the soil profile starting at the surface exists which is wetted to a nearly constant water content and in which capillary pressure is nearly constant. In this region, flow is practically steady and the driving force is mainly gravity. This is called the transmission region.

(2) A transition region exists, below the transmission region, that has a water content decreasing toward the lower end of the region. Within this region, the driving force is predominantly due to a gradient of capillary pressure.

(3) The length of the transition region is relatively longer in soils having a wider range of pore sizes, i.e. , a smaller value of λ.

(4) The value of θ in the transmission region depends on the ratio of q_w / K_m.

(5) The magnitude of K_w in the transmission region approaches q_w.

For cases where q_w at the surface is greater than K_m, the problem is more difficult because a region of θ close to θ_m develops quickly at the surface. Furthermore, the steady inflow rate cannot be maintained. Ponding occurs where $\theta = \theta_m$, and the problem of practical significance is to determine the time at which ponding takes place. The high value of θ near the surface creates a situation such that resistance to air flow may become important.

The latter problem has been studied by Smith (1972), by Mein and Larson (1973) and by Morel-Seytoux (1975). The restricted air-flow case may result from long lateral escape routes, high water tables or soil strata that are impermeable to air because of high water content or dense structure. This case has been investigated by McWhorter (1971) and McWhorter (1976). His formulation of the problem begins with the unsteady 2-phase flow Equations 5.1-5.4, written for the case of 1-dimensional flow.

The equations are combined and converted to the θ-dependent form. It is assumed that air is totally restricted from flowing downward or laterally, so that air can escape only upward through the wetted soil region at a rate equal to the flux rate of water into the soil surface, i.e., $q_{nw} = - q_w$. Other conditions imposed for McWhorter's (1976) solution included $q_w = q_0$ (constant), at $z = 0$ and $t \geq 0$; $\theta = \theta_i$ at $z > 0$ and $t = 0$. McWhorter obtained a solution by a method of successive approximations similar to that used by Parlange (1972) for solving Equation 5.16.

McWhorter (1976) calculated numerical results for his equation as well as that of Parlange (1972) using a Brooks-Corey (1966) relationship among the variables k_w, k_a and p_c, and assuming an arbitrary value of λ and q_0. By comparing the profiles of $\theta = f(z)$ obtained with and without taking air resistance into account, he noted that at early times, air resistance has little effect so that the profiles are nearly the same. At longer times when θ at the surface gets closer to θ_m , the effect of air resistance becomes apparent. At this time, there is a more rapid increase in θ at the surface and ponding may occur shortly after, even when q_0 is less than K_m.

Parlange's (1972) equation does not predict any ponding when q_0 is less than K_m. However, when q_0 is greater than K_m, Parlange's equation predicts a larger time for ponding than is the case with air resistance. On the other hand, McWhorter's (1976) equation probably underestimates the time for ponding, because it assumes that air is incompressible and that $q_0 = q_w$. In a real case, especially where the water table is at a substantial depth, the ponding time may be intermediate between that predicted by the two equations. In the limit, where the depth to the water table is large, Parlange's equation should be accurate; and at small times, both equations should be valid.

The effect of air counter flow has sometimes been observed in field situations, particularly in fields under border irrigation [Dixon and Linden, (1972)]. The magnitude of the effect can be expected to vary with the depth to a water table, the width of borders, the period of time for which the water source persists and on soil properties, among other factors.

The behavior described above for constant rate infiltration, where air resistance is negligible, provides a convenient method for determining K_w as a function of θ or p_c on a wetting cycle. The method depends on the existence of a transmission region at the upper portion of a soil column when water is introduced at a constant rate at the surface. This method has been employed by Davidson et al. (1963) for obtaining $K_w(\theta)$ and by Anat et al. (1965) for obtaining $K_w(p_c)$.

The method employed a procedure similar to that described in Section 4.2.3 for determination of K_w on the drainage cycle by either the long or short-column procedure. The only difference is that the first measurements are made before the column is fully saturated. Water is admitted to the top of the column at a small steady rate. After the wetting front has reached the bottom of the column where the outflow is withdrawn, the upper part of the column is at some constant θ and p_c which are measured. The piezometric gradient within the transmission region is approximately 1.0 and the value of K_w is some constant equal to the inflow rate. The inflow rate is then increased by small increments and a set of measurements is made for each increment until θ approaches θ_m and K_w approaches K_m.

5.2.5. Linear Drainage

Drainage is a process in which air replaces water from soils. A typical field situation of practical interest in this regard is an unconfined aquifer with a falling water table. The flow that occurs in the soil above the water table may be almost linear if the water table is relatively horizontal and remains that way while the water table is lowered. Another case of interest is the redistribution of water toward drier soil below (following a heavy rain or irrigation) where the water table (if one exists) is too deep to affect the flow.

Soil-water engineers, however, are primarily interested in the water table case, because it is for this case that artificial drains may be required. Frequently, artificial drains consisting of perforated tubes are placed in a parallel pattern at some depth below the water table. This creates a $2-$dimensional flow pattern after a rain or irrigation. The water table slopes downward toward the drains creating a horizontal component of flow both below and above the water table.

In the usual field situation, water tables fall gradually toward an equilibrium position following a rain or irrigation. The downward component of the water flux in the soil above the water table may be controlled by the rate at which water can move laterally toward a drain, thus permitting the water table to fall. Although, a sudden incremental lowering of a water table rarely occurs in the field, it often is imposed in laboratory experiments or as a boundary condition for analytical solutions. Consequently, this type of drainage process also has been given considerable theoretical attention. Sometimes a study of drainage following sudden drawdowns leads to insights that permit an understanding of certain field behavior.

The theory used to analyze drainage differs only in details of solution from that employed to study infiltration. Various investigators have idealized the problem as one of single-phase flow of water and have employed the Richards equation in their analyses. More recently the problem has been analyzed as a displacement process involving 2-phase flow [Morel-Seytoux (1975)]. Corey and Corey (1967) employed a scaled physical model to study linear drainage. Their investigations are described in Chapter 6.

Vertical drainage of a soil column (which is initially fully saturated except for trapped air) is a problem of mainly academic interest, but a study of this problem may provide insights into aspects of several real problems. The drainage under consideration occurs when the plane of atmospheric pressure in the soil water is suddenly lowered from one stationary position to another. If the water table is initially at the surface, however, there may be essentially no drainage until the water table is lowered to a depth equal to $p_e/\rho_w g$ below the surface.

The simplest way possible of analyzing this problem is entirely analogous to the Green and Ampt analysis of infiltration. The method might be called an upside-down Green and Ampt approach. It is assumed that an abrupt drainage of the pore space takes place when the pressure of the liquid is lowered below atmospheric pressure to some critical pressure. The critical pressure represents the value of p_e for the soil under consideration. Below the level where p_e exists, the soil is fully saturated and above this point, the soil is assumed to be drained to a water content equal to θ_r. The existence of a transition region is ignored. Ligon et al.(1962) and Youngs (1960) have presented typical analyses based upon this assumption.

The analysis of Youngs (based upon the upside-down Green and Ampt approach) leads to

$$\frac{Q}{Q_\infty} = 1 - e^{-q_0 t/Q_\infty} \qquad 5.26$$

in which Q is the cumulative outflow in time t, Q_∞ is the outflow as t → ∞, and q_0 is the initial flow rate. This equation represents only the first term in a series solution. However, according to Jensen and Hanks (1967), Equation 5.26 is reasonably accurate for Q/Q_∞ up to about 0.6, after which it begins to overestimate the outflow volume.

Gardner (1962) derived another approximate solution for the same boundary conditions. His equation was obtained by assuming that the average conductivity $\overline{K}$ between the surface of a column and the top of the fully saturated region remains constant during the drainage process. He also employed an adjustment (for the increasing length of partially saturated medium) which was proposed by Miller and Elrick (1958). His equation is

$$\frac{Q}{Q_\infty} = 1 - \frac{8}{\pi^2} e^{-\alpha^2 \overline{D} t / L} \qquad 5.27$$

in which $\overline{D} = \overline{K} L^2 / Q_\infty$, L being the length of the column, and α^2 is obtained from the ratio of impedance of the partially saturated region to that of the fully saturated region. At the beginning of drainage, α^2 has the value of $\pi/4$. Equation 5.27 like Equation 5.26, represents the first term of a rapidly converging series.

In comparing the values of Q/Q_∞ predicted by Equation 5.27 with experimental data, Gardner found he could select a value of $\overline{K}$ which would allow the equation to fit the data fairly well. Jensen and Hanks (1967) tried to find a simple way of predetermining $\overline{K}$ because, in a practical case, this would be necessary. The methods they tried for determining $\overline{K}$ did not result in satisfactory agreement between Equation 5.27 and experimental data. They suggested a better method of determining $\overline{K}$ would improve the accuracy of Equation 5.27.

Jensen and Hanks (1967) employed a numerical method developed by Hanks and Bowers (1962) to solve the Richards equation for vertical drainage in one dimension with a plane of atmospheric pressure at the base of a column. They used a computer to obtain Q/Q_∞ as a function of time and pressure as a function of depth and time. Typical results are shown in comparison with experimental data in Figures 5.6 and 5.7.

The functional relationship among S, p_c and k_w needed for the numerical solution of the Richards equation was determined independently by steady-state techniques similar to those described in Section 3.10.3. The results of Jensen and Hanks, therefore, seem to confirm the validity of the Richards equation for an unsteady drainage situation. However, the Richards equation does not rigorously apply to that portion of the draining column which is fully saturated. This is because the function D, equal to $-K\, dp_c/d\theta$

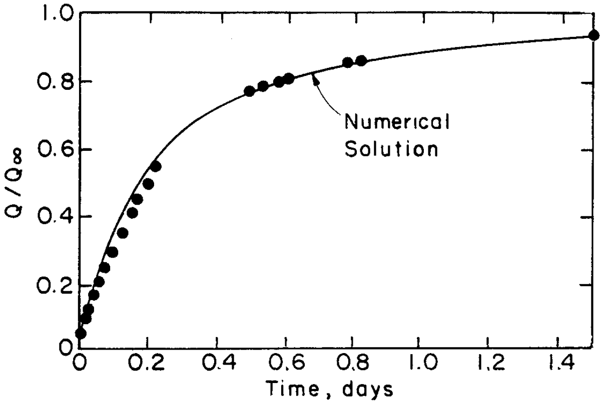

Figure 5.6. Comparison of laboratory data with numerical solution of Richards equation for vertical drainage [Jensen and Hanks (1967)].

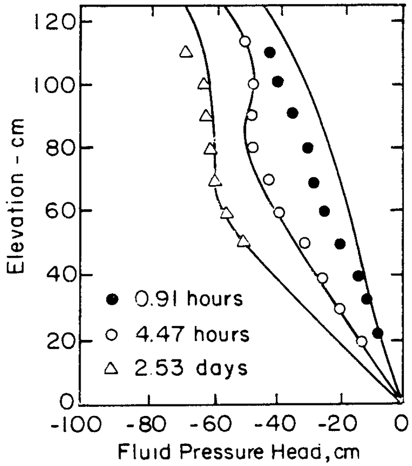

Figure 5.7. Comparison of laboratory data with numerical solution of Richards equation for vertical drainage [Jensen and Hanks (1967)].

does not exist in the fully saturated region. Jensen and Hanks sidestepped this problem by assigning an arbitrarily small constant value to $d\theta/dp_c$ where $p_c < p_e$. They found that the computed drainage rates were insensitive to the exact value of $d\theta/dp_c$ chosen, provided that the value was small.

It would appear from the numerical results of Jensen and Hanks that the 2-phase approach is not necessary in the case of drainage. However, consideration of drainage from the point of view of a 2-phase flow process may provide insights in respect to the interrelation between drainage and infiltration.

5.2.6. Two-dimensional Drainage

A problem of considerable interest to soil-water engineers involves 2-dimensional transient flow toward parallel relief drains. The physical model to be considered is analogous to that described in Section 4.4 except that a period is considered in which no recharge occurs and the water table falls. The problem is to determine the position of the water table at the midpoint between drains as a function of time [see Figure 5.8]. The objective is to determine whether or not a particular depth and spacing of drains satisfies design criteria with respect to lowering of the water table after heavy irrigations or flooding.

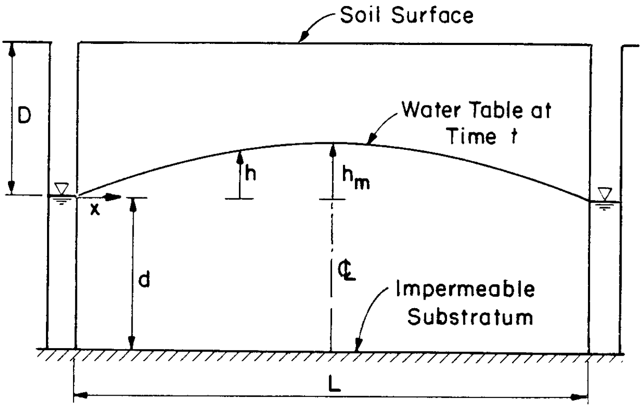

Figure 5.8. Physical model for transient flow towards parallel drains. [McWhorter and Duke (1976)].

The traditional procedure is to ignore both the flow and storage of water above the water table. This approach has led to the well-known and frequently used equation of Glover which has been described by Dumm (1964 and 1967). Use is made of the Dupuit-Forchheimer assumptions for flow in the region below the water table; that is, it assumes the vertical component of flow is negligible. This assumption leads to

$$K_m \frac{\partial}{\partial x}\left[(h + d)\frac{\partial h}{\partial x}\right] = \phi_e \frac{\partial h}{\partial t}, \qquad 5.28$$

a result which is reasonably accurate if d is large compared to h and if the region of flow above the water table is small compared to d. Equation 5.28 is known as the Dupuit-Forchheimer equation.

Equation 5.28 is nonlinear because the coefficient of $\partial h / \partial x$ includes h. Glover linearized this equation by replacing h + d by an average value of $h + d$ designated as $\bar{d}$. Therefore, Glover's equation has the form

$$\alpha \frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t}, \qquad 5.29$$

where

$$\alpha \equiv \frac{K_m \bar{d}}{\phi_e}.$$

This equation provides satisfactory results (where Equation 5.28 is also valid) provided the slope of the water table is very small so that linearization does not introduce a significant error.

In many cases, however, $p_e / \rho_w g$ is not small compared to $\bar{d}$ so that it is not sufficient to neglect either flow or storage above the water table, or to assume that the specific yield S_y is a constant equal to ϕ_e. Theoretically, the problem can be solved rigorously treating the system as a 2-phase flow region and solving the unsteady 2-phase flow Equations 5.1-5.4. For the $2-$dimensional transient case under consideration, a rigorous solution requires a complex numerical solution.

McWhorter and Duke (1976) have presented an approximate analytical solution which accounts for flow and storage above the water table. Their solution also accounts (to a large extent) for the nonlinearity of the flow system.

A differential slice of soil, oriented parallel to the axis of the drains is shown in Figure 5.9. The depth below the soil surface to the top of the saturated region is denoted by z_d and the thickness of the fully saturated

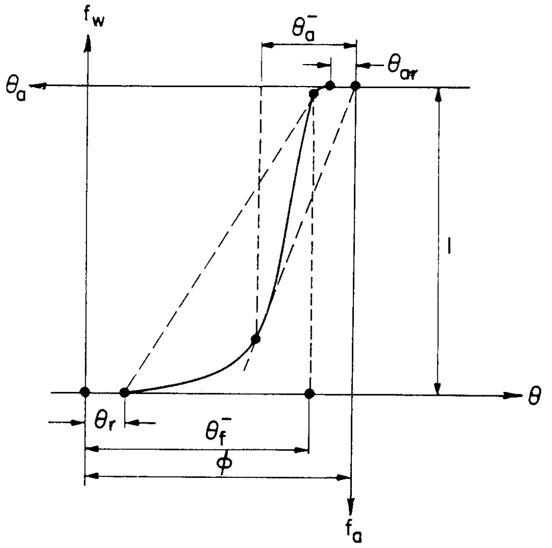

Figure 5.9. **Differential slice of soil, oriented parallel to axis of drains [McWhorter and Duke (1976)].**

region (except for trapped air) above the water table is ψ_d. H_k is the effective permeable height used by Duke (1973) for the analysis of steady flow towards parallel drains [Section 4.4]. H_k is a height of saturated soil having the same capacity to transmit water as does the actual soil region above the water table. The mathematical expression for H_k is given by

$$H_k \equiv \frac{1}{K_m}\int_0^H K_w dz \; .$$

With this addition to the flow section, Equation 5.28 becomes

$$K_m \frac{\partial}{\partial x}\left[\left(h + d + H_k\right)\frac{\partial h}{\partial x}\right] = \frac{\partial V}{\partial t} \qquad 5.30$$

in which V is the volume of drainable water per unit surface area stored in the slice at any time.

An approximate expression for V is obtained using the assumption that the distribution of water in the slice is as if the vertical component of flow is negligible. This is a reasonable assumption since in field situations the rate of lowering of the water table is extremely slow. The empirical expression for effective water saturation of Brooks and Corey (1966),

$$S_e = \left(\frac{z_d - z + \psi_d}{\psi_d}\right)^{-\lambda},$$

is used to evaluate water content at any point above the fully saturated region. Integration of $\phi_e (1 - S_e)\,dz$ from the top of the fully saturated region to the soil surface ($z = 0$) and differentiation of the result with respect to time yields

$$\frac{\partial V}{\partial t} = \phi_e \psi_d \left[1 - \left(\hat{z}_d + 1\right)^{-\lambda}\right] \frac{\partial \hat{z}_d}{\partial t} \qquad 5.31$$

in which $\hat{z}_d = z_d / \psi_d$. Equation 5.31 is a tractable continuity equation which accounts for the major effects of storage above the water table.

Substituting Equation 5.31 into Equation 5.30, scaling all quantities having a dimension of length by ψ_d and noting that $\hat{z}_d = \hat{D} - \hat{h} - 1$ results in

$$\frac{K_m}{\phi_e \psi_d} \frac{\partial}{\partial \hat{x}} \left[\left(\hat{h} + \hat{d} + \hat{H}_K\right) \frac{\partial \hat{h}}{\partial \hat{x}}\right] = \left[1 - \left(\hat{D} - \hat{h}\right)^{-\lambda}\right] \frac{\partial \hat{h}}{\partial t}. \qquad 5.32$$

For a static water distribution, the Brooks-Corey relationship for $K_w(p_c)$ gives

$$\hat{H}_k = 1 + \frac{1}{1 + 3\lambda} \left[1 - \left(\frac{1}{\hat{D} - \hat{h}}\right)^{1+3\lambda}\right].$$

In cases of practical interest (where the water table is not very near the soil surface), $\hat{h} << \hat{D}$, so that

$$\hat{H}_k \approx 1 + \frac{1}{1 + 3\lambda} \left[1 - \left(\frac{1}{\hat{D}}\right)^{1+3\lambda}\right]. \qquad 5.33$$

In this analysis, Equation 5.33 is used to evaluate $\hat{H}_k$. In most cases of practical interest $\hat{H}_k < 1.4$. Note that in this formulation $\hat{H}_k$ is treated as a constant, its dependence on $\hat{h}$ being considered negligible.

Expanding the derivative on the left side of Equation 5.32 gives

$$\frac{\partial}{\partial \hat{x}}\left[\left(\hat{h}+\hat{d}+\hat{H}_K\right)\frac{\partial \hat{h}}{\partial \hat{x}}\right] = \left(\hat{h}+\hat{d}+\hat{H}_k\right)\frac{\partial^2 \hat{h}}{\partial \hat{x}^2}+\left(\frac{\partial \hat{h}}{\partial \hat{x}}\right)^2 .$$

Except in the immediate vicinity of the drains at small times, the second term on the right is negligible compared to the first term on the right, so that

$$\alpha\left(\hat{h}\right)\ \frac{\partial^2 \hat{h}}{\partial \hat{x}^2} = \frac{\partial \hat{h}}{\partial \hat{t}}, \qquad 5.34$$

in which

$$\alpha\left(\hat{h}\right) \equiv \frac{K_m}{\phi_e\,\psi_d}\left[\frac{\hat{h}+\hat{d}+\hat{H}_k}{1-\left(\hat{D}-\hat{h}\right)^{-\lambda}}\right].$$

Equation 5.34 retains the important effects of the nonlinearity of flow. The conditions imposed on Equation 5.34 are expressed as

$$\hat{h}\left(0,\hat{t}\right) = \hat{h}\left(\hat{L},\hat{t}\right) = 0 .$$

This condition ignores the existence of the seepage face at the drain wall. However, Equation 5.34 implies that the entire interval $\hat{H}_k$ is available for flow, including that part where the water is at sub-atmospheric pressure. The latter simplification should tend to compensate in some degree for ignoring the seepage face.

Duke (1973) obtained a finite difference solution for Equation 5.34. Later, McWhorter and Duke (1976) found an approximate analytical solution for this equation. Figure 5.10 shows some typical results from their solutions compared with experimental data obtained by Hedstrom et al. (1971) with a large soil flume. Results from the solution of Glover's linear Equation 5.29 also are shown in this figure. In Figure 5.10, h_m is the water table elevation above the drains at the midpoint and H_0 is the initial water table elevation [see Figure 5.10]. Equation 5.34 clearly provides a better approximation to the measured data than does Equation 5.29 for this case.

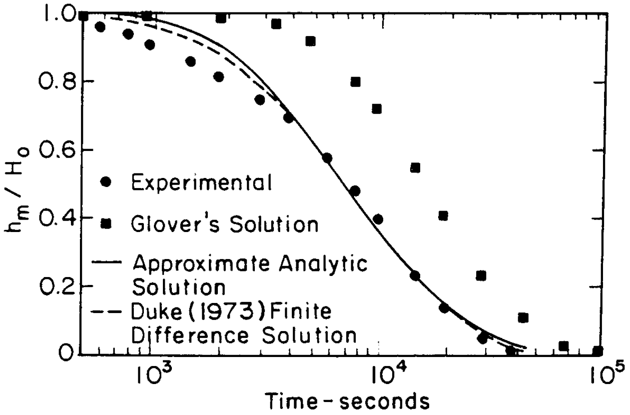

Figure 5.10. Drawdown at midpoint between parallel drains as a function of time [McWhorter and Duke (1976)].

5.3. HORIZONTAL LINEAR IMBIBITION WITH NON-WETTING PHASE RESISTANCE

The problem analyzed here is that presented in Section 5.2.1, but full consideration is given to the viscosity of the nonwetting phase. The results are rigorously correct for systems with two incompressible fluids. Also, some insights into the effects of air viscosity in air-water systems are obtained, even though air is highly compressible.

Equations 5.1, 5.2 and 5.4 are combined to yield an expression for the wetting-phase flux that incorporates the viscous resistance to flow of both phases:

$$q_w = f q_t - D \frac{\partial S}{\partial x} \quad . \qquad 5.35$$

The quantity q_t appearing in Equation 5.35 is referred to as the total flux

$$q_t \equiv q_w + q_{nw}$$

and is, at most, a function of time as can be deduced immediately from Equation 5.3. Because q_w and q_{nw} may have different signs and magnitudes, the total flux can be zero, positive, or negative depending upon the problem under consideration. The analysis in this section is restricted to imbibition of the wetting phase, in which case $q_t \geq 0$.

The functions f and D are sensitive functions of S and are defined by

$$\mathbf{f} \equiv \left(1 + \frac{\mu_w \, k_{nw}}{\mu_{nw} \, k_w}\right)^{-1}$$

and

$$\mathbf{D} \equiv -\frac{k_{nw} \, f}{\mu_{nw}} \frac{dp_c}{dS} .$$

The function D has the dimensions of a diffusivity and is a generalized version of the diffusivity defined in Section 5.2.1. When the nonwetting fluid viscosity approaches zero, the two D functions approach the same quantity. However, the reader should note that the two-phase diffusivity of this section is greatly different from the diffusivity used when the viscosity of the nonwetting phase is assumed zero. The function f has no counterpart in the traditional treatment of air-water systems.

A key variable in the mathematical analysis of the imbibition problem is the function F_n defined by

$$F_n \equiv \frac{q_w - f_i \, q_t}{(1 - f_i) q_t}$$

where $f_i = f(S_i)$ and S_i is the initial saturation (also, the saturation at $x = \infty$). This function is a normalized version of the fraction of total flux contributed by the wetting phase.

It has been shown [McWhorter (1971); McWhorter and Sunada (1990) and (1992)] that F_n is a function only of S for the imbibition problem and can be calculated by iteration in

$$F_n = F_o \left[1 - \frac{\displaystyle\int_S^{S_m} \frac{(\beta - S) D}{F_n - f_n} d\beta}{\displaystyle\int_{S_i}^{S_m} \frac{(S - S_i) D}{F_n - f_n} dS} \right] \qquad 5.36$$

where f_n is a normalized f given by

$$f_n \equiv \frac{f - f_i}{1 - f_i} \, .$$

The saturation on the plane $x = 0$ has been taken to be S_m, where S_m is the maximum achievable saturation under wetting conditions. To achieve a boundary saturation other than S_m requires that the wetting phase be introduced through a membrane that is impervious to the nonwetting fluid.

The quantity F_o in Equation 5.36 is $F_n(S_m)$, the value of the fractional-flow function at $x = 0$. For imbibition problems, $F_o \geq 1$ with the actual value determined by the wetting-phase pressure drop imposed across the saturation profile. The implications of $F_o \geq 1$ and the conditions under which it arises are the most interesting aspects of two-phase imbibition but are more easily elucidated after the formal solution is complete.

The saturation profile is given by McWhorter and Sunada (1990),

$$x(S,t) = \frac{2A(1 - f_i)}{\phi} \frac{dF_n}{dS} t^{1/2} \, , \tag{5.37}$$

where A is a constant given by McWhorter and Sunada (1990),

$$A^2 \equiv \frac{\phi}{2(1 - f_i)^2 F_o} \int_{S_i}^{S_m} \frac{(S - S_i) D}{F_n - f_n} \, dS \, .$$

The required derivative dF_n/dS is calculated from Equation 5.36 to be

$$\frac{dF_n}{dS} = F_o \frac{\displaystyle\int_{S}^{S_m} \frac{D}{F_n - f_n} d\beta}{\displaystyle\int_{S_i}^{S_m} \frac{(S - S_i) D}{F_n - f_n} dS} \, . \tag{5.38}$$

The physical significance of the constant A is revealed by its presence in the flux equation

$$q_t = A t^{-1/2} \tag{5.39}$$

where it is seen to be the determining factor for the total flux.

At this point an algorithm for the computations emerges. A value for F_o is arbitrarily selected, subject to $F_o \geq 1$, and $F_n(S)$ is computed from Equation 5.36. With $F_n(S)$ now known, A is computed from its definition. This determines the total flux as shown in Equation 5.39. Finally, dF_n/dS is computed from Equation 5.38 and the saturation profile is calculated from Equation 5.37. Thus, it is observed that specification of F_o determines all other features of the problem.

The differences between imbibition processes calculated with different values of F_o are shown in Figures 5.11 and 5.12. Figure 5.11 shows $F_n(S)$ for three different values of F_o. At all saturations for which $F_n(S)$ is less than unity, the wetting and nonwetting phases flow in the same direction. This is deduced from the definition of $F_n(S)$. The saturation for which $F_n(S) = 1$ is the point at which the nonwetting fluid flux is zero. At all points where $F_n(S) > 1$, counter-current flow of nonwetting phase occurs. Thus, it is concluded that for any $F_o > 1$, there is a zone in the vicinity of $x = 0$ in which counter-flow of the nonwetting fluid is established. Beyond this zone, the two phases flow co-currently. This interesting feature of the two-phase imbibition problem was first established by Chen et al. (1990) and described briefly by McWhorter and Sunada (1990).

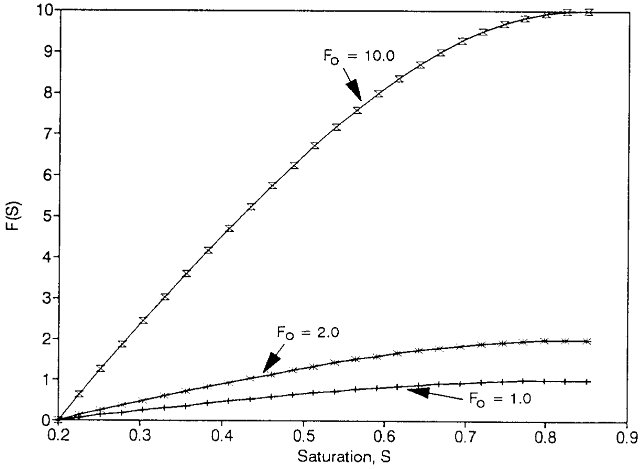

Figure 5.11. F(S) for three values of F_o.

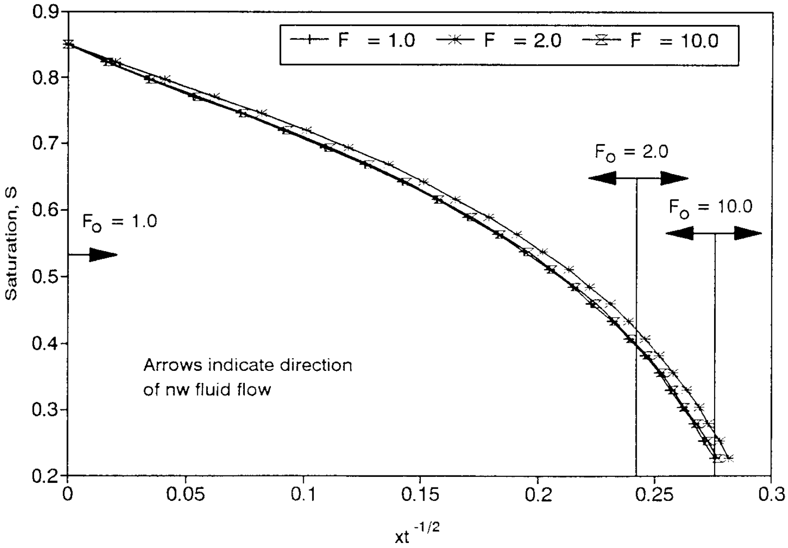

Figure 5.12. Co-current and counter-current flow divide .

The flow divide separating the co-current and counter-current flow zones is shown on the imbibition profiles in Figure 5.12. The larger the value of F_0, the greater the extent of the counter flow zone. As $F_0 \rightarrow \infty$, counter flow occurs throughout. On the other hand, the flow is everywhere co-current for $F_0 = 1$. The value of the flux parameter A appearing in Equation 5.39 declines with increasing F_0 and becomes zero for $F_0 \rightarrow \infty$.

The results of the foregoing analysis are correct only if the input flux is indeed proportional to $t^{-1/2}$. An obvious way to create the imbibition flows of this section is to impose such a flux as a boundary condition by pumping the wetting fluid into the column according to

$$q_w(o,t) = F_o A t^{-1/2} . \qquad 5.40$$

The linkage between a particular imbibition process and the physical conditions required to produce the process, say in the laboratory, could be established by selecting a value of F_0, computing the corresponding value of "A", and then set the pump according to Equation 5.40. The experimental set-up must allow the nonwetting phase to exit freely from both ends of the sample while the wetting fluid is injected at x = o, according to Equation 5.40. Computer controlled syringe pumps are available that can closely approximate the rate given by Equation 5.40.

The counter flowing nonwetting fluid exiting the column into the end chamber must take on the pressure of the fluid inside the chamber and external to the porous medium. This implies a capillary pressure of zero at $x=0$. However this contradicts the requirement that capillary pressure must equal or exceed the entry pressure before the nonwetting fluid exists as a continuous phase. In fact, the nonwetting fluid exits in a sporadic manner as the capillary pressure fluctuates rapidly between zero and the entry pressure, as explained in Section 4.3.

An examination of Equation 5.37 shows that "A" is a maximum when $F_o = 1.0$. Since for the imbibition process $F_o \geq 1.0$, the product, F_o A, must have a finite maximum, A_m, equal to "A" calculated for $F_o = 1$. However, the pump used to inject the wetting fluid is not constrained. If the pump injects wetting fluid at a rate exceeding that consistent with the assumed boundary conditions, the plane at which $S = S_m$ can no longer remain stationary at x = 0. This plane is propagated into the porous medium, forming an expanding region in which the saturation is constant and equal to S_m. In this case, Equation 5.36 for $F_n(S)$ must be replaced by a more general form presented by McWhorter and Sunada (1990).

Consider a column of finite length L, initially containing wetting fluid at a uniform saturation, $S_i < S_m$. Let the nonwetting fluid be at atmospheric pressure. One end of the column (x = 0) is suddenly exposed to wetting fluid at atmospheric pressure, while the wetting fluid pressure at x = L is controlled at the initial value, so the pressure drop in the wetting fluid across the column is constant. This is the way imbibition experiments are usually conducted for air-water systems.

Application of the theory presented in this section shows that the constant pressure drop imposed in the experiment is inconsistent with Equation 5.39. In other words, the above experiment conducted with incompressible and viscous fluids, would not result in an imbibition rate proportional to $t^{-1/2}$. Such an experiment would yield square-root-of time behavior only if the pressure drop in the nonwetting fluid, downstream of the wetted portion, is zero. The small viscosity and high compressibility of air contribute to minimal pressure drop in the air downstream of the wetting front in air-water experiments. In addition, laboratory experiments are often conducted so that displaced air can escape through perforations in the column walls, further minimizing the resistance to air flow downstream of the wetting front. As a result, water imbibition proportional to $t^{-1/2}$ is often observed.

In addition to the insights into the physics of imbibition described above, the solution of Equation 5.35 provides a tool for evaluating a variety of simulation models that have been proposed for predicting two-phase flow in porous media. This is because Equation 5.35 is exact when applied to a linear horizontal displacement process, provided both the displacing and displaced phases are incompressible viscous fluids.

5.4. DISPLACEMENT OF PETROLEUM FLUIDS

Petroleum fluids, e.g., crude oil and natural gas, are produced from porous rocks, usually aquifers containing brine, by displacement of one fluid by another. The displacement process may involve the displacement of oil by brine from surrounding rock or from injection wells, or it may involve the displacement of oil by natural gas evolving from solution or invading from an expanding gas cap. In such systems, brine is usually the wetting phase and oil is the nonwetting phase. When gas is present, it is the nonwetting phase in respect to either oil or brine. The displacement process may occur spontaneously when a well is drilled into an aquifer, or it may be induced by an artificial injection of either brine or gas.

In some cases, fluids may be produced from an aquifer simply by compression of the pore space upon the reduction of fluid pressure or by the expansion of the fluids. The matrix compression and fluid expansion are significant mainly when they take place over a very large volume of the aquifer, and the effect locally (at the site of a producing well) is mostly the replacement of petroleum fluids by fluids being squeezed out of the surrounding aquifer.

The theory used in analyzing petroleum displacements is basically the same as that for air-water or LNAPL-water displacements, since each involves the displacement of one immiscible fluid by another. Consequently, the starting equations are the same, i.e., Equations 5.1 - 5.4. However, the relative magnitudes of the several forces producing the flow are often quite different. Therefore, the simplifications made in the case of petroleum displacements are likely to be very different from those made for air-water or LNAPL-water displacements, as explained in Section 5.1.

5.4.1. Linear Displacement

The displacement of one fluid by another in a 3-dimensional system is difficult to analyze mathematically. Fortunately, considerable insight into the practical problems encountered can be obtained by analyzing 1 – dimensional flow systems. Furthermore, many field situations can be approximated by assuming 1 – dimensional models as an idealization of the real situation [Richardson (1961)].

Most oil producing aquifers are thought to have been originally saturated with brine that subsequently was partly replaced by oil and perhaps gas, although brine remains as a residual wetting phase. The question to be answered is how much oil can be produced when a well is drilled into the formation before the operation of the well is no longer economical. Operation of a well becomes uneconomical some time after the invading brine breaks through to the well, and the production of brine becomes large compared to the production of oil. Often, the brine comes from a line of injection wells and the brine advances along a broad wetting front toward a line of producing wells.

Such a system may sometimes be idealized as a linear displacement process. A long thin tube of homogeneous porous material is used to model the displacement process. The tube may be inclined at an angle β to a horizontal plane. The porous material is originally fully saturated with brine, but the brine (in excess of residual saturation) is replaced later by oil. The tube of porous material is then in a condition called the restored state, presumed to be similar to the condition of many petroleum aquifers before petroleum production begins.

Subsequently, some of the oil is replaced by brine by flooding the tube from one end where the brine is introduced at high pressure. A question to be answered is how are the fluids distributed along the tube at any time during the flooding process. Another question is how much oil is produced from the outflow end of the tube as a function of time and as a function of the injected volume of brine.

An analysis of the idealized case provides some insights into the practical problem. Formulation of a flow equation for this case involves simplification of Equations 5.1-5.4. Most petroleum aquifers have a small slope, and the displacement takes place under large pressure gradients over large distances where the fluids flow side by side with small saturation gradients. Therefore, it is reasonable to drop the gravity terms in Equations 5.1 and 5.2, and to assume one pressure gradient applies to both fluids.

Furthermore, the idealized case is one of linear displacement so that the directional subscripts also can be dropped from the governing equations. With these simplifications, Equations 5.1-5.3 can be combined to give

$$\phi\frac{\partial S}{\partial t} = -\frac{\partial}{\partial x}\left[q_{nw}\left(\frac{\mu_{nw}\,k_w}{\mu_w\,k_{nw}}\right)\right] . \qquad 5.41$$

Under the condition of a negligible capillary pressure gradient, the term on the right of Equation 5.41 is an approximation of $-\partial q_w / \partial x$. However, q_w varies with x so q_w is a varying fraction F_w of the total flux rate $q_w + q_{nw}$, designated as q_t. The fraction F_w is assumed to be a function of S only, for a given pair of fluids. Equation 5.41 is rewritten in terms of q_t and F_w as

$$\phi\,\frac{\partial S}{\partial t} = -\frac{\partial}{\partial x}\left(q_t\,F_w\right) .$$

The assumption that the fluids are incompressible is equivalent to assuming that q_t does not vary with x so that

$$\phi\,\frac{\partial S}{\partial t} = -q_t\,\frac{\partial F_w}{\partial x} .$$

Since F_w is presumed to be a function of S only,

$$\frac{\partial S}{\partial t} = -\left(\frac{q_t}{\phi}\,\frac{dF_w}{dS}\right)\frac{\partial S}{\partial x} . \qquad 5.42$$

The saturation S is a function of x and t only, where x is regarded as the coordinate of a plane (perpendicular to the tube) where S has any specified value. Therefore, the total derivative of S with respect to time is given by

$$\frac{dS}{dt} = \frac{\partial S}{\partial x}\frac{dx}{dt} + \frac{\partial S}{\partial t} .$$

The quantity dx/dt is interpreted as the velocity of a plane (perpendicular to the tube) having any specified value of S, i.e., dx/dt = $(dx/dt)_S$. Since $(dS/dt)_S = 0$,

$$\left(\frac{dx}{dt}\right)_S = -\frac{\partial S}{\partial t}\Big/\frac{\partial S}{\partial x} .$$

Combining this with Equation 5.42 gives

$$\left(\frac{dx}{dt}\right)_S = \frac{q_t}{\phi}\left(\frac{dF_w}{dS}\right)_S , \qquad 5.43$$

which is called the Buckley-Leverett equation [Buckley, S. E. and Leverett, M. C. (1942)].

If the permeability ratio k_{nw}/k_w as a function of S is known, dF_w/dS can be evaluated as a function of S. When S is known at $t = 0$, Equation 5.43 can be integrated to give the saturation distribution for any $t > 0$. Unfortunately, particular values of dF_w/dS may occur at two different values of S, so that two different values of S are calculated for particular values of $(x)_S$, a physical impossibility. This problem is resolved by postulating a discontinuity in S at a front, and requiring that a material balance be satisfied. The volume of displacing fluid behind the front must equal the volume of fluid displaced.

An explanation of this procedure is presented in a paper by Buckley and Leverett (1942) and in texts on petroleum engineering, including one by Collins (1961). However, the above procedure can give only a rough approximation of the real distribution of fluids near the front because the effect of capillarity is substantial in this region and this effect is neglected in Equation 5.43.

The integration of Equation 5.43 with respect to time gives

$$(x)_S = \frac{Q_i(t)}{\phi A}\left(\frac{dF_w}{dS}\right)_S , \qquad 5.44$$

in which $Q_i(t)$ is the cumulative total volume of fluid flowing through an area A in an interval of time t, and $(x)_S$ is the distance traveled along x by the plane of saturation S in the same interval of time. Experience has shown that Equation 5.44 gives a good approximation of the actual distribution of S for long systems with large flow rates, except near the advancing front as previously noted.

5.4.2. Determination of Permeability Ratios Using the Welge Technique

When conditions are such as to justify the use of Equation 5.43, it can be integrated using a method proposed by Welge (1952). The following analysis applies to the displacement of oil by brine from a horizontal tube of homogeneous porous material which has been brought to a restored state as described in Section 5.4.1. In this case, the tube has a length L, area A and a zero angle of inclination. At time t = 0, brine is introduced at x = 0 at high pressure. This causes brine to flood the tube at a rate q_w which is a function of time.

At x = 0, only brine is flowing at t > 0, while at x = L, only oil flows initially. After breakthrough of brine, both fluids flow at x = L. Designating the cumulative inflow by Q_i, the cumulative outflow of oil by Q_o (and brine by Q_w, it is noted that $Q_i = Q_o + Q_w$.

At the outflow end,

$$\mathbf{F_o = \frac{dQ_o}{dQ_i}},$$

where F_0 is the fraction of the total volume flux that consists of oil flux. For the conditions under consideration, $F_o \approx f_{nw}$, where f_{nw} is an approximation of F_{nw} valid when both fluids flow under nearly the same potential gradient. In this case,

$$\mathbf{q_{nw} \approx \frac{k_{nw}}{\mu_{nw}} \left(\frac{\partial p^*}{\partial x}\right)}$$

and

$$\mathbf{q_t \approx \left(\frac{k_{nw}}{\mu_{nw}} + \frac{k_w}{\mu_w}\right) \left(\frac{\partial p^*}{\partial x}\right)},$$

where p^* is the piezometric potential. Consequently,

$$\mathbf{\frac{q_{nw}}{q_t} \approx \left(1 + \frac{\mu_{nw} k_w}{\mu_w k_{nw}}\right)^{-1} = f_{nw}}\ .$$

An analogous approximation of F_w is designated as f_w.

The approximation for F_0 leads to

$$\mathbf{\left(1 + \frac{\mu_{nw} k_w}{\mu_w k_{nw}}\right)^{-1} \approx \frac{dQ_o}{dQ_i}}\ .$$

Solving the latter equation explicitly for the permeability ratio k_w/k_{nw} gives

$$\frac{k_w}{k_o} = \frac{\mu_w}{\mu_o}\left(\frac{dQ_i}{dQ_o} - 1\right) . \qquad 5.45$$

Consequently, the permeability ratio at any t > 0 can be determined from the ratio of brine inflow to oil outflow rates. The ratio is zero until brine breakthrough after which it gradually increases and eventually becomes infinitely large when the outflow ceases completely.

The value of S at the outflow end of the tube can be computed as a function of Q_i and dQ_i/dQ_o by solving a material balance equation; i.e.

$$\phi\int_o^L (1 - S)dx = \phi L\,(1 - S_r) - (Q_o / A) ,$$

in which A is the cross-sectional area of the tube and L is its length. Integrating by parts gives

$$\phi L(1 - S_L) - \phi\int_{S_o}^{S_1} x(-dS) = \phi L(1 - S_r) - (Q_o / A),$$

where S_L is S at x = L. Note that (-dS) is inserted in the integral because this is equivalent to d(l-S). The equation may be rewritten as

$$S_L = S_r + \frac{Q_o}{A\phi L} + \frac{1}{L}\int_{S_o}^{S_L} x\, ds . \qquad 5.46$$

It was pointed out by Welge (1952) that the sum of the first two terms on the right of this equation gives the average wetting phase saturation of the tube at any time during the flood.

According to Equation 5.44, the third term on the right of the expression for S_L can be written as

$$\frac{Q_i}{A\phi L}\int_{x=o}^{x=L} dF_w ,$$

or

$$-\frac{Q_i}{A\phi L}\int_{x=o}^{x=L} df_{nw} .$$

At the inflow end, $f_{nw} = 0$, and at the outflow end, $f_{nw} = dQ_0/dQ_i$. Therefore,

$$\mathbf{S_L = S_r + \frac{Q_o}{A\phi L} - \frac{Q_i}{A\phi L}\left(\frac{dQ_o}{dQ_i}\right)_L} \; . \qquad 5.47$$

All quantities on the right of Equation 5.47 can be determined from inflow and outflow data as is the case for Equation 5.45.

Equation 5.45 and 5.47 provide a method of determining permeability ratios as a function of S on small rock cores. The method is widely used in the petroleum industry. Cores of rock from producing formations are obtained with a core drill at the time wells are drilled. Such cores are usually about 10 cm in diameter. In order to obtain cores parallel to the bedding, smaller cores are removed from the well cores with the axis of the smaller cores being more or less parallel to the bedding. The latter cores are usually about 2 to 2.5 cm in diameter and about 5 to 8 cm in length.

The residual fluids are extracted from the laboratory cores and the porosities are determined. The surface of the cores (except for the ends) are sealed, usually with an acrylic plastic. The cores are then saturated with brine. Next, all of the brine except residual brine is replaced by oil. The core is then in the restored state as previously described.

Finally, brine is injected at one end of the core at a very high pressure, and the outflow of both oil and brine from the opposite end are measured separately until the outflow of oil nearly stops. The cumulative outflow of oil Q_o is plotted as a function of the total outflow Q_i. The slope of the curve, dQ_o/dQ_i is substituted into Equations 5.45 and 5.47 to obtain k_w/k_o as a function of S.

Equations 5.45 and 5.47 apply strictly for long tubes only. However, it is believed that the error introduced by their application to short cores is minimized by using larger pressure gradients. The Buckley-Leverett equation also is used to describe displacement of oil by gas from short cores.

In this case, where the displacing fluid is a gas, the error is likely to be more serious. However, the error is thought to be substantially reduced by first saturating the core with an oil of very high viscosity which permits displacement experiments to be carried out under an extremely large pressure gradient.

In any case, this technique is widely used in the petroleum industry to obtain k_g/k_o ratios as a function of S. The ratios k_g/k_o and k_o/k_w are used in the petroleum industry to predict reservoir behavior as explained in Section 5.4.5. Both of these ratios tend to plot as a straight line on a semi-log plot of k_{nw}/k_w as a function of S as shown in Figure 5.13.

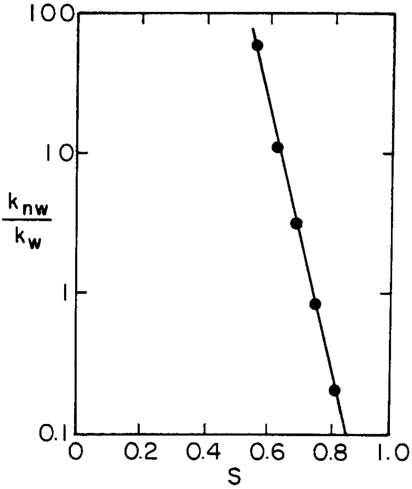

Figure 5.13. Permeability ratio as a function of saturation.

5.4.3. Calculation of Relative Permeabilities from Linear Displacement Experiments

For some calculations, the determination of k_{nw}/k_w curves is not sufficient and individual relative permeability curves for each flowing fluid are needed as well. Johnson et al. (1959) have shown that linear displacement experiments on small cores can be used to determine k_{rnw} and k_{rw} as functions of S.

The analysis upon which the method of Johnson et al. is based makes use of several relationships that can be deduced from the Buckley-Leverett and Welge theories. For a water-oil system, these are:

(1) $\mathbf{f_o q_t = -(k_o/\mu_o)(\delta p/\delta x)}$, which is Darcy's equation for the oil phase.

(2) $\mathbf{f_w/f_o = k_w \mu_o / k_o \mu_w}$, which can be deduced from the definitions of f_w and f_{nw}.

(3) $\mathbf{S_{av} = (S)_L + (Q_i / A\phi L)(f_o)_L}$, where the subscript L indicates that the quantities are evaluated at the outflow end of the sample. This is the Welge Equation 5.46.

(4) $\mathbf{(dX/dt)_S = (q_t/\phi)(dF_w/dS)_S}$, the Buckley-Leverett equation.

(5) $\mathbf{(x)_S = (Q_i/\phi A)(dF_w/dS)_S}$, obtained by integrating the Buckley-Leverett equation over an interval of time and distance of travel of the saturation S.

(6) $\mathbf{Q_i / A\phi L = 1/(df_w/dS)_L}$, which comes from evaluating the relationship (5) at the outflow end where $(x)_S = L$. Note that $Q_i / A\phi L$ represents the number of pore volumes injected.

(7) $\mathbf{(f_o)_L = A\phi L(dS_{av}/dQ_i)}$, which is obtained by differentiating the relationship (3) with respect to Q_i. Note that for a fixed value of $(f_o)_L$, S_L is a constant.

The sample preparation and experimental procedure is the same as for the determination of k_o/k_w curves described in Section 5.4.2.

At a given instant during the displacement experiment, the pressure drop across a sample of length L is given by

$$\Delta p = \int_o^L \frac{\delta p}{\delta x}\, dx \, ,$$

or from (1)

$$\Delta p = -q_t \mu_o \int_o^L \frac{f_o}{k_o}\, dx \, . \qquad 5.48$$

According to (5) and (6), the various saturations (at a particular instant) occur at distances which can be calculated from

$$\frac{(x)_S}{L} = \frac{(df_w/dS)_S}{(df_w/dS)_L} \, ,$$

or

$$x = \frac{(f_w^*)_S}{(f_w^*)_L} L \, ,$$

and

$$dx = \frac{L}{(f_w^*)_L} df_w^*$$

in which $(f_w^*)_L$ has some fixed value depending upon the particular instant under consideration. Substituting this expression for dx into Equation 5.48 gives

$$\frac{\Delta p (f_w^*)_L}{q_t \mu_o L} = \int_o^{(f_w^*)_L} \frac{f_o}{k_o} df_w^* .$$

Substituting the product $k_{ro}\,k_m$ for k_o gives

$$\frac{\Delta p\, k_m (f_w^*)_L}{q_t \mu_o L} = \int_o^{(f_w^*)_L} \left(\frac{f_o}{k_{ro}}\right) df_w^* . \qquad 5.49$$

According to Johnson et al. (1959), the quantity $q_t \mu_o L / \Delta p k_m$ has been called relative injectivity I_r by Rapoport. I_r and $(f_w^*)_L$ are interrelated since both vary with the amount of water injected. Differentiating both sides of Equation 5.61 with respect to $(f_w^*)_L$ results in

$$\frac{d[(f_w^*)_L / I_r]}{d(f_w^*)_L} = \left(\frac{f_o}{k_{ro}}\right)_L .$$

Substituting the relationship (6) into the latter expression gives

$$\frac{d\left(\frac{1}{Q_i\, I_r}\right)}{d\left(\frac{1}{Q_i}\right)} = \left(\frac{f_o}{k_{ro}}\right)_L . \qquad 5.50$$

During the experiment, Q_i and Q_o are recorded as well as Δp. This permits the evaluation of the derivatives in Equation 5.50 and in the relationship (7). From (7), $(f_o)_L$ is determined permitting the evaluation of $(k_{ro})_L$. The corresponding value of $(S)_L$ is determined from (3), S_{av} being given by

$$S_{av} = S_r + \frac{Q_o}{A\phi L} .$$

5.4.4. Calculation of Relative Permeabilities by Application of Linear Scaling

A displacement resulting from the injection of a single fluid is said to be linearly scalable if the average saturation of any portion of a porous sample is a function only of the number of pore volumes of displacing fluid that have been injected into or through the portion under consideration [Parsons, R. W. and Jones, S. C. (1977)]. The processes discussed in Section 5.4 are linearly scalable if the assumptions made concerning these processes are valid. In fact, Equation 5.44 implies that the displacement is linearly scalable.

Jones and Roszelle (1978) have shown how the acceptance of the principle of linear scalability can simplify the mathematics and expedite the calculations involved in the determination of relative permeabilities from displacement experiments on laboratory samples. To understand their procedures, it is necessary first to examine the concept involved in specifying the number of pore volumes injected.

The concept can be examined in reference to Figure 5.l4.

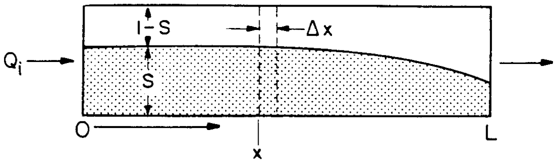

Figure 5.l4. Schematic of sample undergoing linear displacement of oil by water.

In Figure 5.l4, Q_i indicates a volume of water that has been injected through an area A at time t (through the input face of a linear sample initially saturated with oil and residual water). The height of the shaded area indicates the water saturation S at any vertical plane along the length of the sample. The sample has a length L, and the coordinate x indicates a distance along the sample beginning at x = 0. The number of pore volumes injected PV_i for the entire sample at any time is $Q_i/\phi AL$, whereas PV_i for the portion extending from 0 to x is a larger number given by $Q_i/\phi Ax$. Note that as $x \rightarrow 0,\ PV_i \rightarrow \infty$, even for a small value of Q_i.

The water saturation increases and the oil saturation decreases as PV_i increases. The value of S reaches some limiting value when $PV_i \rightarrow \infty$. Theoretically, the limiting (maximum) value of S is obtained instantly (upon the beginning of injection) at the input surface where x = 0. The value of S decreases with x and is smallest at x = L. However, when Q_i becomes very large, the variation of S with x is very small, and as Q → ∞, is theoretically at the limiting value at all x.

A mathematical advantage is gained by starting with the assumptions that:

(1) Saturation and other properties related to S are single-valued functions of PV_i only.

(2) The same functional relationship between any of these properties and PV_i holds for values of the variables averaged over a segment of a homogeneous sample regardless of the size of the segment, provided the segment is large enough to be a representative element of the porous medium.

It is possible to show that any property satisfying the condition of being a single-valued function of S will also be a single-valued function of PV_i only. In fact, the theory of Buckley and Leverett (1942) and Welge (1952) imply this is the case provided the relative permeabilities are single-valued functions of S.

To demonstrate the mathematical consequences of linear scaling, an unspecified property P is considered which satisfies the linear scaling conditions. This property might be S or some function of S, e.g., f_w, k_{rw}, or k_{ro}. The following analysis also can be applied to any function of the latter variables, for example, it also applies to P^{-1}. In general, P varies with x and Q_i in a cylindrical porous sample during a displacement process in which the displacing fluid is injected through a face perpendicular to x at x = 0. The value of P at x for a particular Q_i is defined by

$$P_x = \lim \Delta x \rightarrow 0 \left[\frac{(x + \Delta x)\,\overline{P}_{(x+\Delta x)} - x\overline{P}_x}{\Delta x} \right],$$

or

$$P_x = \lim \Delta x \rightarrow 0 \left[\overline{P}_{(x+\Delta x)} + \frac{x(\overline{P}_{(x+\Delta x)} - \overline{P}_x)}{\Delta x} \right],$$

in which $\overline{P}_x$ is the average value of P over the segment extending from 0 to x. Therefore, from the definition of a derivative

$$P_x = \overline{P}_x + x\frac{\partial \overline{P}_x}{\partial x},$$

in which all quantities including the derivative are functions of PV_i. Since $\overline{P}_x$ is assumed to be a single-valued function of PV_i only, this also can be written as

$$P_x = \overline{P}_x + x\left[\frac{d\overline{P}_x}{d(PV_i)}\frac{\partial(PV_i)}{\partial x}\right]_x .$$

Because PV_i is $Q_i/\phi Ax$,

$$\frac{\partial(PV_i)}{\partial x} = -\frac{q_i}{\phi A x^2}$$

so that

$$P_x = \overline{P}_x - (PV_i)_x\left[\frac{d\overline{P}_x}{d(PV_{i)}}\right]_x . \qquad 5.51$$

Applying Equation 5.51 to S_x at $x = L$, and noting that the increase in S_{av} of the sample as a whole is related to the outflow of oil Q_o, gives

$$S_L = S_{av} - \frac{Q_i}{\phi AL}\left(\frac{dQ_o}{dQ_i}\right)_L ,$$

which is Equation 5.48. Jones and Roszelle (1978) have presented a convenient graphical procedure for evaluating S_L using plots of S_{av} as a function of Q_i from laboratory displacement data. An expression for $(k_{rw})_L$ corresponding to S_L also can be derived using the linear scaling principle. The procedure employs the concept of an effective viscosity (or relative reciprocal mobility Λ_r^{-1}) which is defined so that a flow equation for q_t is

$$q_t = -\frac{k}{\Lambda_r^{-1}}\left(\frac{dp}{dx}\right)_x \qquad 5.52$$

in which k is the permeability when the sample is fully saturated with water and $(dp/dx)_x$ is assumed to apply for either the water or the oil phase. By writing separate equations for q_w and q_o and adding these to obtain q_t it can be shown that

$$\Lambda_r = \left(\frac{k_{rw}}{\mu_w} + \frac{k_{ro}}{\mu_o}\right) = \Lambda/k , \qquad 5.53$$

which is called relative mobility, Λ being called total mobility. A second equation for q_t is given by

$$\left(f_w\right)_x q_t = -\frac{k\,k_{rw}}{\mu_w}\left(\frac{dp}{dx}\right)_x .$$

Equating the latter expression to Equation 5.53 and solving for k at the outflow end gives

$$\left(k_{rw}\right)_L = \left(f_w\right)_L \mu_w \left(\Lambda_r\right)_L . \tag{5.54}$$

Since k_{rw} and k_{ro} are considered to be single-valued functions of S, it follows that Λ_r and Λ_r^{-1} are also single-valued functions of PV_i. Therefore, Equation 5.51 can be applied to Λ_r^{-1}, i.e.,

$$\left(\Lambda_r^{-1}\right)_L = \left(\Lambda_r^{-1}\right)_{av} - \frac{Q_i}{\phi AL}\left[\frac{d(\Lambda_r^{-1})_{av}}{d(PV_i)}\right]_L . \tag{5.55}$$

Jones and Roszelle (1978) have described a graphical procedure for solving Equation 5.55 similar to that used by them to obtain S_L. However, the data needed are values of $(\Lambda_r^{-1})_{av}$ as a function of PV_i. Values of $(\Lambda_r^{-1})_{av}$ are obtained as a function of PV_i from the experimental data which include Δp across the entire sample; i.e.,

$$\left(\Lambda_r^{-1}\right)_{av} = \frac{k}{q_t}\,\frac{\Delta p}{L} \tag{5.56}$$

which results from integrating Equation 5.64 over x from 0 to L.

The linear scaling principle and the procedures employed by Jones and Roszelle involve the same theoretical limitations as the procedures for Johnson et al. (1959). These include the requirement that dp/dx is very large compared to dp_c/dx and that k_{rw} and k_{ro} are single-valued functions of S only.

The latter assumption may not be entirely valid since there is evidence that both k_{rw} and k_{ro} are to some extent dependent on q_t. This has been observed particularly in respect to the residual oil saturation during water floods, the residual oil saturation supposedly representing the saturation at which k_{ro} is zero. However, increasing the injection rate decreases the residual oil saturation slightly. Lowering the oil-water interfacial tension also reduces the residual oil saturation. Furthermore, injection rates necessary to minimize the effect of dp_c/dx (in small laboratory samples) are much greater than those found in field applications. It is believed, however, that this difficulty is not very serious because there is evidence that k_{rw} and k_{ro} are not very sensitive functions of rate.

5.4.5. Prediction of Reservoir Behavior from Permeability Ratios

Permeability ratios and relative permeabilities as functions of saturation constitute the basic data used in the prediction of petroleum reservoir behavior. Equations 5.1-5.4 provide the theory for the calculations. The processes to be evaluated may involve water displacing either oil or gas (or a gas displacing oil). Sometimes more than one of these processes occur simultaneously. The displacements are rarely truly linear, but they often can be idealized as such with an error that is not large compared to other uncertainties involved.

Linear displacements are often described by the steady two-phase flow equations presented in Section 4.1.2. However, it is convenient for many purposes to rearrange Equation 4.2, and to write it explicitly for the ratio q_w/q_t designated as F_w. The result is

$$\mathbf{F_w = \frac{1 + \dfrac{k_{nw}}{q_t\,\mu_{nw}}\left(\dfrac{\partial p_c}{\partial x} + \Delta\rho g_i\right)}{1 + \dfrac{k_{nw}\,\mu_w}{k_w\,\mu_{nw}}}} \ . \qquad 5.57$$

Equation 5.57 is called the fractional flow equation, and has an important role in reservoir engineering calculations. When both $\partial p_c / \partial x$ and $\Delta\rho g_i$ are negligible, $F_w \approx f_w$ as noted in Section 5.4.1. The latter situation frequently prevails for petroleum displacements, especially where brine displaces oil. However, it may sometimes be desirable to retain the term $\Delta\rho g_i$ where oil is displaced by gas. The retention of $\partial p_c / \partial x$ is rarely required in routine petroleum engineering calculations because field systems are usually long enough that this term is negligible.

In most cases which can be idealized as linear, Equation 5.44 (or a modification of it) can be used to predict reservoir behavior. For a displacement by water (brine), for example, it is often useful to predict the amount of oil that can be recovered before the water-oil ratio at the producing well becomes such that additional production is uneconomical.

The data needed, in addition to k_o/k_w curves, include the thickness of the oil-bearing stratum, the distance from the wetting front to the producing wells, the values of S_r, ϕ, k_m, μ_w, μ_o and the water injection rate Q_i/t per unit width of stratum. If the term $\Delta\rho g_i$ is to be included in the expression for F_w the densities ρ_o and ρ_w and the angle of inclination of the formation must also be included.

The first step in the computation is the determination of F_w, which for the case of small $\Delta\rho$ and/or small angles of inclination is approximated by

$$f_w \equiv \left[1 + \left(k_o \mu_w / k_w \mu_o\right)\right]^{-1}.$$

A plot of $f_w(S)$ is made and the slope df_w/dS is measured at intervals of S (say at increments of 0.05), beginning with some value of S greater than S_r. Typical $f_w(S)$ and (df_w/dS) (S) functions are shown in Figure 5.15.

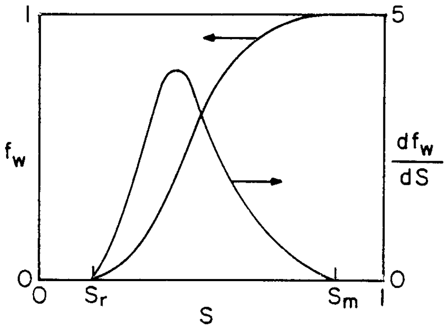

Figure 5.15. The fractional flow of wetting fluid and its derivative as a function of saturation.

Values of df_w/dS are recorded for the selected values of S. The distances travelled by the various saturations during a given time interval are determined from Equation 5.44, approximated by

$$(x)_s = \frac{Q_i}{\phi A}\left(\frac{df_w}{dS}\right)_s,$$

in which Q_i is the volume (per unit width of stratum) of brine injected during the same interval, and A is the thickness of the stratum.

A plot of S(x) is made, as illustrated in Figure 5.16, for a number of time intervals. The curve identified as t1 indicates that, in this time interval, the flood front has not yet reached the producing wells. Note, that before the time of breakthrough, the area under the curve (above S_r) equals $Q_i / \phi LA$, because the volume of oil displaced must equal the volume of brine injected. After breakthrough, the area under the curve equals the volume of oil displaced but, in general, this is less than the volume of brine injected because the wells produce brine as well as oil.

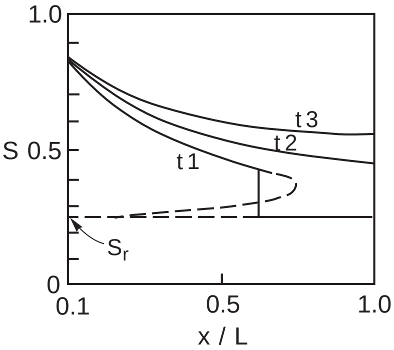

Figure 5.16. Saturation profiles for a linear displacement

Equation 5.44 does not permit an explicit solution of the maximum distance travelled by the flood front. In fact, the location of a given calculated saturation is double-valued, because df_w/dS is double-valued as explained in Section 5.4.1. The dotted line in Figure 5.16 illustrates this situation. However, the real saturation is not double-valued since a portion of the curve is not applicable to the displacement process. A discontinuity in saturation at the front is postulated such that the area under the curve (above S_r) behind the discontinuity equals $Q_i / \phi LA$. In an actual case, the saturation is not discontinuous at the front, the apparent discontinuity being a consequence of neglecting $\partial p_c / \partial x$ in the expression for F_w.

The Q_i for breakthrough can be determined by trial from plots as shown in Figure 5.16, and if Q_i/t is known, the time for breakthrough is also determined. After breakthrough, the value of F_w gradually increases at the producing wells where x = L. The amount of oil that can be recovered by the time F_w reaches a specified value can be determined from a version of Equation 5.47 in the form

$$\left(1 - F_w\right)_L = \frac{Q_o / A\phi L - (s)_L + S_r}{Q_i / A\phi L} .$$

As an example, consider the curve designated as t2 in Figure 5.16. This curve has been plotted for a given time interval corresponding to a specified Q_i and a specified value of $(S)_L$. The value of $Q_o / A\phi L$ is given by the area under the curve above S_r. The specified values of Q_i and $(S)_L$ along with the computed value of $Q_o / A\phi L$ permits the calculation of F_w for this particular value of t and Q_i. The process is repeated for a number of values of Q_i. A plot of Q_o as a function of Q_i typically has characteristics as illustrated in Figure 5.17, in which Q_b indicates the water injected at the time of breakthrough.

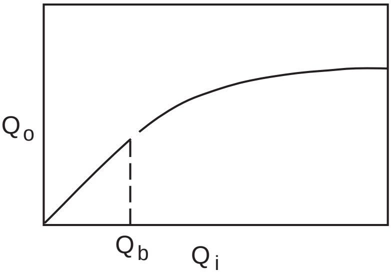

Figure 5.17. Oil recovered as a function of water injected for a linear displacement.

Since the ultimate recovery of oil is approached asymptotically, it is never economically feasible to reach the ultimate recovery. The amount of water that must be produced becomes excessive before the ultimate recovery is achieved.

PROBLEMS AND STUDY QUESTIONS

1. When formulating problems involving displacement of one fluid by another, petroleum reservoir engineers frequently state that they have ignored capillary effects. Explain in what sense they have, in fact, ignored capillary effects and under what conditions this is a valid approximation.

2. The integrated form of the Buckley-Leverett equation (with the term $\partial p_c / \partial x$ omitted) gives a reasonably accurate approximation of the actual distribution of S for long systems with high flow rates. Explain why the length of a system is significant in this respect.

3. What difficulties would be expected in using the technique of Johnson et al. (based on the Welge integration of the Buckley-Leverett equation) for determining air and water relative permeabilities of soil materials.

4. Starting with the 2-phase unsteady flow equations, show that the fact (of air viscosity being much smaller than water viscosity) is not sufficient to justify neglecting air resistance in a water-air displacement process.

5. In a soil water-air system, one should not assume F_w is approximately equal to f_w. Explain.

6. Explain why the diffusivity equation is not valid for the description of flow in a fully saturated soil.

7. According to King (1964), the vertical dimension of a horizontal tube of soil (used to check the square root of time law) should be small compared to $p_d/\rho_w g$. Describe some difficulties that would be expected if this condition were not satisfied.

8. Describe an important mathematical advantage of writing a flux equation for an infiltration analysis in terms of the total flux.

9. Would the same mathematical advantage apply for an infiltration process involving 2- or 3-dimensional flow? Explain.

10. Explain why the fractional-flow analysis employed by Morel-Seytoux for infiltration problems is more effective for determining infiltration rates than for determining saturation distributions.

11. Explain why a flux equation in the form of the Green and Ampt flux equation can be exact, whereas an equation combining this flux equation with a continuity equation is usually not exact.

12. When infiltrometers are used to measure infiltration rates in a field overlying a high water table, it is possible that the measured rates may be significantly higher than those observed during a flooding operation. Explain.

13. Describe a set of conditions under which infiltration may be described by the "diffusivity" equation with negligible error.

14. Describe a set of conditions under which it is more appropriate to use a 2-phase flow analysis of the infiltration process.

15. Consider an experiment in which a small square plot of ground, say four square meters, is exposed to a small constant rate input of water (from a sprinkler) slightly less than K_m. Assuming that a water table exists at a depth of 2 meters, would the 2-phase flow equations of McWhorter, accounting for air compression, give a better evaluation of ponding time than the solution of Parlange for the diffusivity equation? Explain.

16. Consider a case of drainage from an initially fully saturated column of sand. The water table is suddenly lowered to the base of the column at time $t = 0$. Assuming the water table is maintained at this depth throughout the period of observation, derive an equation (based on an upside down Green and Ampt analysis) for the discharge rate as a function of the cumulative discharge at a particular time.

17. Discuss the possible consequences of analyzing drainage for the case described in problem 16 by assuming the distribution of water above the point where $p_c = p_e$ to be a static distribution. Would you expect this assumption to provide a more or less accurate description of the discharge rate as a function of the cumulative discharge than that obtained from the upside down Green and Ampt analysis? Explain.

Chapter 6

MODEL SIMILARITY

6.1. MODELS

Only a small fraction of systems encountered in practice are amenable to analytical solutions. More often, boundary conditions are too complex to be expressed in simple mathematical terms. In such cases, it is necessary to idealize the problem or to obtain solutions by using a finite difference or finite element model, and perhaps to check the results with an appropriate physical model.

An appropriate model, whether physical or numerical, should create the same relationship among the pertinent variables as the system being analyzed, i.e., the prototype. In other words, the model should be similar to the prototype. Numerical models may require considerable time to formulate and substantial expense for computer time. Consequently, it is desirable to obtain results in a form applicable to as large a range of field conditions as practical. To accomplish this, the governing equations (and the initial and boundary conditions) should be expressed in terms of variables that minimize the requirements for similarity. This is accomplished by choosing the most appropriate set of variables to describe the system and scaling the variables with appropriate scale factors.

Physical models are more expensive to construct and operate than numerical models. Therefore, it is desirable to determine the least restrictive criteria of similarity so that the model is as inexpensive to construct and operate, as well as being similar to as large a range of field situations as possible.

6.2. DEFINITION OF SIMILARITY

The technical meaning of "similarity" is somewhat more restricted than its non-technical meaning, because it refers to a specified closed set of variables and not to every conceivable property. For example, variables describing a fluid system might include pressure, density and velocity, but not color. Two systems are similar in a technical sense when the same relationship exists among the specified variables after they have been transformed in an appropriate manner.

The size of a set of similar systems depends in part upon what variables have been selected to describe the systems, and in part upon how the variables are transformed. An objective of the transformation process is to maximize the set of similar systems. The process is perhaps more easily understood with respect to geometric variables. In any case, an understanding of geometric similarity is essential before other kinds of similarity can be explained.

6.2.1. Geometric Similarity

When investigating similarity of physical systems, boundaries enclosing the systems must be considered. Spaces enclosed by boundaries of the same shape are geometrically similar regardless of their size. For example, every sphere is similar to every other sphere regardless of the length of its radius. Likewise every cube is similar to every other cube regardless of the dimension of its sides.

On the other hand, a prolate spheroid (generated by rotating an ellipse about its major axis) is not necessarily similar to other prolate spheroids. The shape of such a space depends upon the ratio of two characteristic dimensions, i.e., the lengths of the major and minor axes. If the ratios of the two axes are equal, the spheroids are similar. Furthermore, if two prolate spheroids are similar, the ratio of lengths of the major axes are the same as the ratio of lengths of the minor axes.

The length of one of the axes (say the minor axis) can be selected as a common "scale factor" with which to measure the "scaled length" of any other "corresponding dimension," e.g., the major axis. If prolate spheroids are similar, the scaled lengths of corresponding dimensions are equal.

An analogous relationship can be used to define geometric similarity of spaces with boundaries of any shape, regardless of their conformation or symmetry. However, an infinite set of dimensions is needed to specify the shape in the general case. Furthermore, the meaning of the term "corresponding" in the general case is not as obvious as for prolate spheroids.

An origin for a coordinate frame of reference can be located within space enclosed by the boundaries of any system. If spaces are similar, it is possible to identify a common point for an origin in each of the spaces, say at the centroids. Coordinate axes can be oriented so distances to the boundary from the origin divided by a common "scale factor" are equal in any particular direction. Coordinate frames of reference consistent with this condition are called corresponding frames of reference. Similar geometric spaces are those for which corresponding frames of reference exist.

A geometric scale factor is a characteristic dimension of spaces being compared. For example, a suitable scale factor could be the length of a line segment from the origin of a corresponding frame of reference to the boundary in a particular direction. With such a scale factor, the scaled distance to the boundary from the origin in any given direction is the same for all geometrically similar spaces.

Points located at equal scaled distances in the same direction from the origin of corresponding frames of reference are called corresponding points. Lengths of line segments connecting corresponding points are called corresponding dimensions. Any corresponding dimension is a valid scale factor, because it characterizes the size of spaces being compared.

There is an infinite set of factors consistent with this definition of a geometric scale factor. In choosing a particular factor from the set, it is appropriate to select one that can be identified without ambiguity and which can be measured easily with adequate precision.

Similarity of geometry in respect to three coordinate directions may not be required in some models. For example, models of flow systems characterized adequately as 1-dimensional vertical flow problems might be small-diameter columns, only the depth dimension in relation to media properties being important. Usually, it is necessary to maintain geometric similarity only for dimensions included in differential equations describing the interrelation among the variables. This is one way in which an appropriate selection of variables can make requirements for similarity less restrictive.

When investigating geometric similarity for some systems, similarity of more than one kind of boundary must be considered. For example, when dealing with two fluids in a porous matrix, it is necessary to consider (at least implicitly) the geometry of a particular fluid with boundaries consisting of internal solid surfaces and fluid interfaces, as well as the external boundary of the porous matrix. The need to consider the internal geometry explicitly on a microscopic scale may sometimes be avoided by selecting "macroscopic" variables to describe systems, where the latter variables are volume averages of corresponding "microscopic" variables. This is another way an appropriate selection of variables can make requirements for similarity less restrictive.

When boundaries of a system are changing with time, a more restricted definition of the characteristic dimension used as a scale factor may be necessary. For example, it may be necessary to define the scale factor as the length of a line segment between two corresponding points at a particular scaled time. The question of how time is scaled is discussed in the following section.

An example of a system for which the boundaries change with time is unsteady flow in an open channel. Similarity with respect to a geometric variable in this case implies the system is geometrically similar at any particular scaled time. However, spaces enclosed by boundaries at some particular clock time are not similar unless the systems are identical. For example, geometries of two similar flow systems in open channels (in which traveling waves exist) are not similar when compared at a common clock time unless they are identical channels with identical boundaries and identical conditions. Otherwise, the geometries will be similar only at a common scaled time.

6.2.2. Similarity of Non-geometric Variables

Similarity in respect to a non-geometric variable implies the scaled values of the variable are equal at corresponding points at corresponding times. This definition applies only for systems remaining geometrically similar at any scaled time, because the definition is in reference to conditions at corresponding points, which exist only in geometrically similar spaces.

Corresponding times are times having equal scaled values. Similarity in respect to time means equal values of other scaled variables occur at corresponding points at equal scaled times. However, scale factors used for variables must be characteristic parameters of the system.

6.2.3. Characteristic Parameters

As explained in Section 6.2.1, a characteristic length parameter is the length of a line segment connecting any two corresponding points in a corresponding frame of reference. An infinite set of suitable scale factors exist for length. However, if a particular length scale factor is a characteristic parameter of the system, other suitable scale factors for length differ only by a common multiplier. For example, the radius of a pipe is a characteristic length dimension for pipes and the internal diameter is also a characteristic length dimension for pipes, because it is always twice the radius. However, the outer diameter of a pipe does not characterize the fluid system because its magnitude is not related by a common multiplier to the radius of the flowing column of fluid. Use of the latter quantity for scaling merely changes the unit of measurement and serves no useful purpose.

Vectors differ from scalars such as length in that they are assigned directions as well as magnitude. For example, position, velocity and acceleration are quantities characterized by direction as well as magnitude. However, only magnitudes are scaled. Two scaled vectors are equal if their directions as well as their scaled magnitudes are equal.

Variables may be classified also according to whether or not they are assigned values at particular points. A variable which varies with respect to position in a system is called an intensive variable. The vectors mentioned above are examples of intensive variables. Pressure and temperature are examples of intensive variables that are scalar quantities.

Similarity with respect to intensive variables requires distributions of the variable in both space and time to be similar. Any non-zero magnitude of an intensive variable at a corresponding point at a corresponding time is a suitable scale factor. For example, if velocity vectors during steady flow in a tube with circular cross sections are scaled with the magnitude of the velocities at the centerline, the scaled velocity vectors are equal at equal radii in all tubes undergoing laminar flow. This is because the direction of flow at all radii is parallel to the axis, and scaled magnitudes of velocities are equal at corresponding points. Average velocities Q/A in tubes can also be used as velocity scale factors because they are directly proportional to velocities at the centerline for all tubes with similar velocity distributions. Velocity at any corresponding radius, except at boundaries, could also be used as scale factors. Velocity at boundaries can not be used, because it has a zero magnitude.

6.2.4. Explicit and Derived Scale Factors

If x_o is a characteristic length and u_o is a characteristic velocity, x_o/u_o is a characteristic time parameter for a system. Furthermore, any length parameter directly proportional to x_o/u_o also is a valid time scale factor. Parameters x_o and u_o are examples of "explicit" scale factors whereas x_o/u_o is an example of a "derived" scale factor. Derived scale factors are combinations of other scale factors and constant system parameters having appropriate dimensions.

The number of explicit scale factors required is not greater than the number of basic dimensions needed to express all variables in a given set. For most flow systems, the number of basic dimensions is not more than three, i.e., force, length and time. Often the number is less than this. Explicit scale factors are used to scale variables for which valid scale factors are clearly indicated. For example, the depth of water in an open channel or the diameter of a pipe is selected to scale lengths, and the maximum permeability of a porous medium is selected as an explicit scale factor for effective permeabilities. Variables for which no obvious characteristic parameter exists are scaled with derived scale factors.

6.3. CRITERIA OF SIMILARITY

6.3.1. Selection of Variables

Establishing criteria of similarity for any system begins with the identification of a set of variables describing aspects of the system of interest to the investigator. Systems often can be viewed from more than one perspective, depending on the kind of reference elements to which the variables refer, and consequently alternative sets of variables may be selected. Normally, sets are selected for which criteria of similarity are least restrictive and for which criteria can most easily be established.

For example, flow of two fluids in a porous matrix may be considered from either a "microscopic" or a "macroscopic" viewpoint. With the former viewpoint, it is necessary to consider force and material balances on microscopic reference elements of each individual fluid. Moreover, boundaries of each fluid consist partly of internal solid surfaces and partly of changing interfaces between the two fluids.

With a macroscopic viewpoint, reference elements are macroscopic volume elements of porous matrix, so force and material balances refer to volume averages over macroscopic elements. Characteristics of internal geometry are accounted for implicitly, through functional relationships between fluid content, permeability and pressure difference across interfaces.

Inevitably, a macroscopic viewpoint results in a shorter list of variables and system parameters and a more easily established set of similarity requirements. Describing a particular process in greater detail, or in greater generality than required usually results in criteria of similarity more restrictive than necessary.

6.3.2. Inspectional Analysis

Criteria of similarity can be developed from a set of variables by dimensional considerations alone [Buckingham (1914)]. The process is called "dimensional analysis." However, criteria can be developed more directly, and usually in a more useful form, by scaling the governing equations.

Similar systems are those for which scaled governing equations and conditions yield identical particular solutions. It is usually possible to determine (by inspection of the scaled equations) conditions that yield identical particular solutions. The procedure is often called "inspectional analysis."

6.3.3. Similarity for Steady Flow of a Single Fluid in Porous Media

Steady flow of a single liquid in a fully saturated porous material is considered to illustrate the method of inspectional analysis. The first step is choosing an appropriate set of variables. In this case a macroscopic viewpoint is indicated and the variables are q_i, p, h and x_i. Elevation, h, is a function of x_i and is not an independent variable. The relationship among the variables is assumed to be given by a form of Darcy's equation, i.e.,

$$q_i = -\frac{k}{\mu}\left[\frac{\partial p}{\partial x_i} + \gamma \frac{\partial h}{\partial x_i}\right]. \qquad 6.1$$

In this equation k is the permeability to the fully saturated liquid and μ is the liquid viscosity. Specific weight of the liquid is designated by γ.

This form of Darcy's equation assumes media to be isotropic and fully saturated so k is a constant scalar. The equation applies only to a region where $p > -p_e$ so p_e is an obvious choice as a scale factor for p. Flux variables q_i apply to reference volumes of porous matrix rather than to fluid particles, but have dimensions of velocity. Since scale factors for flux and space are not immediately apparent, q_o and x_o are derived. The first version of the scaled equation is

$$q_o \hat{q}_i = -\frac{k}{\mu}\left[\frac{p_e}{x_o}\frac{\partial \hat{p}}{\partial \hat{x}_i} + \gamma \frac{\partial \hat{h}}{\partial \hat{x}_i}\right] \qquad 6.2$$

where scaled variables are indicated by a hat.

Each variable scaled in an equation requires the scale factor to be introduced in a constant coefficient of the operator, in which the variable appears, to avoid altering the equation. Derived scale factors are defined to set as many constant coefficients as possible equal to unity to minimize requirements for similarity. Each constant coefficient not set equal to unity introduces a criterion of similarity, i.e., each corresponding coefficient must be equal for equations to yield identical particular solutions for different systems.

The constant coefficients can be removed from the brackets on the right side of Equation 6.2 by defining

$$\mathbf{x_o} \equiv \frac{\mathbf{p_e}}{\boldsymbol{\gamma}} .$$

This scale factor represents the maximum height of a column that could remain fully saturated with a static liquid when the top and bottom are exposed to a common ambient gas pressure. The remaining constant coefficient is set equal to unity by defining

$$\mathbf{q_o} \equiv \left[\frac{\mathbf{k}\boldsymbol{\gamma}}{\boldsymbol{\mu}}\right] .$$

This scale factor represents the flux rate corresponding to a hydraulic gradient of unity. The final scaled version of Darcy's equation is

$$\mathbf{\hat{q}_i} = -\left[\frac{\partial \mathbf{\hat{p}}}{\partial \mathbf{\hat{x}_i}} + \frac{\partial \mathbf{\hat{h}}}{\partial \mathbf{\hat{x}_i}}\right] . \qquad 6.3$$

Equation 6.3 is identical to Equation 6.1 although constant parameters in Equation 6.1 have been incorporated in the scaled variables. Both are valid for the same range of physical conditions and are subject to the same limitations. For example, they are valid only for isotropic media, and media in which $\hat{p} > -1$. Smaller values of pressure allow ambient gas to enter and desaturate the media.

Equation 6.3 refers to small reference elements of porous matrix and does not assume homogeneous media. However, if media properties vary from point to point Equation 6.3 is insufficient to describe the flow and additional relationships, describing the variation of media properties in space, are required.

Equation 6.3 is inspected to determine criteria of similarity for steady flow in media that are homogeneous as well as isotropic and contain a single homogeneous liquid. Conditions are deduced permitting the equation to yield identical particular solutions in terms of scaled variables. Since all constant coefficients have been set equal to unity, the criteria of similarity are minimized. They are:

1. Boundary conditions in terms of scaled variables must be the same. Boundary conditions include the size and shape of space occupied by the porous matrix.

2. The orientation of matrix boundaries with the gravitational field must be the same.

If the variables had not been scaled, identical solutions would require porous systems to be identical in size as well as similar in shape. Equal permeabilities, viscosities and specific weights also would be required. Clearly generality is enhanced by scaling, and the class of systems to which one can apply results of a laboratory experiment or data from a numerical model is expanded. Moreover, the basis upon which data from a model can be related to behavior of a prototype is identified.

Fluid mechanics textbooks provide an example of similarity analysis for flow of a single incompressible Newtonian viscous fluid in enclosed conduits. For such cases, a microscopic viewpoint is indicated and reference elements are "fluid particles." The Navier-Stokes equation is assumed to describe relationships among the variables. Three constant coefficients appear in the scaled equation which can not be set equal to unity, i.e., the Froude, Euler, and Reynolds numbers raised to some power. Equality of these numbers are included in the requirements for similarity.

The validity of Darcy's equation requires viscous force to predominate over both fluid inertia and gravity in determining the scaled velocity distribution at the pore scale. Consequently, none of the coefficients mentioned appear in the scaled version of Darcy's equation.

Texts dealing with open channel flow show how criteria of similarity for fully developed turbulent flow can be minimized by considering reference elements consisting of sections of channels rather than fluid particles. In this case one is not concerned with the distribution of velocities at the microscopic scale, but gravity and varying depth of flow are significant factors. The scaled version of the open channel flow equation does not have coefficients involving Euler or Reynolds numbers but does have coefficients involving the Froude number.

6.4. SIMILARITY FOR FLOW OF TWO FLUIDS IN POROUS MEDIA

Since flow in porous media involves Newtonian viscous fluids undergoing negligible divergence, the criteria of similarity deduced from the Navier-Stokes equation theoretically should be applicable in this case. In applying these criteria, however, several complexities of 2-phase flow must be considered. One complication is that the Navier-Stokes equation must be applied separately to each of the fluid phases so the criteria of similarity deduced from it for each phase must be combined.

Furthermore, boundaries of flow systems pertinent to solutions of the Navier-Stokes equation are internal solid surfaces of pore spaces and interfaces between two fluids. Microscopic boundaries, therefore, vary with saturation. Saturation varies with capillary pressure and is influenced by surface tension and by wettability as well as by pore geometry.

Microscopic analyses result in criteria which are very difficult to satisfy and may, in fact, be mutually incompatible. However, many early investigators of similarity for porous media included microscopic variables in their analyses [Leverett et al. (1942)]. Furthermore, most early investigators deduced criteria of similarity by "dimensional" rather than "inspectional" analyses, and consequently did not arrive at criteria of similarity in the most useful form.

The Navier-Stokes equation can be simplified by dropping the acceleration term. This is a reasonable simplification since fluid inertia is rarely a significant factor in 2-phase flow in porous media. Dropping the acceleration term eliminates Reynolds number from similarity criteria. A further simplification can be made by noting that density occurs in the remaining terms only as the product ρg so that the two variables can be treated as one, as was pointed out by Loomis and Crowell (1964).

Miller and Miller (1956) were the first to investigate similarity specifically for air-water systems in soils. Their approach leads to a concept of "similar media" which requires the internal pore geometry to be similar, i.e., to differ only by a constant scale factor. When media are similar in this sense they also have equal porosities.

Brooks and Corey (1966) presented a set of criteria without explicit reference to any pore-scale variables. Their approach is entirely macroscopic, proceeding directly from the Richards equation. This theory was later extended by McWhorter and Corey (1967) to include systems for which flow of nonwetting fluids cannot be ignored. They scaled a 2-phase flow equation obtained by combining a set of governing equations. However, some of the variables appearing in the set of governing equations do not appear explicitly in the combined equation. As a result, one criterion of similarity was overlooked by the McWhorter-Corey (1967) procedure, i.e., equality of ratios of specific weights of the two fluids.

6.4.1. Governing Equations

The governing equations for flow of two fluids in a porous matrix are:

1. Darcy's equation for a wetting fluid:

$$q_{wi} = -\frac{k_w}{\mu_w}\left[\frac{\partial p_w}{\partial x_i} + \gamma_w \frac{\partial h}{\partial x_i}\right] . \qquad 6.4$$

2. Darcy's equation for a nonwetting fluid:

$$q_{nwi} = -\frac{k_{nw}}{\mu_{nw}}\left[\frac{\partial p_{nw}}{\partial x_i} + \gamma_{nw} \frac{\partial h}{\partial x_i}\right] . \qquad 6.5$$

3. Material balance equations:

$$\phi \frac{\partial S}{\partial t} = -\frac{\partial q_{wi}}{\partial x_i} = \frac{\partial q_{nwi}}{\partial x_i} . \qquad 6.6$$

4. Functional relationships:

$$k_w,\ k_{nw},\ S = f(p_{c)} . \qquad 6.7$$

5. From the definition of capillary pressure:

$$\frac{\partial p_c}{\partial x_i} = \frac{\partial p_{nw}}{\partial x_i} - \frac{\partial p_w}{\partial x_i} . \qquad 6.8$$

6.4.2. Limitations of Governing Equations

The governing equations have the following limitations:

1. Convection processes only are described -- not diffusion.
2. Media are isotropic, so permeabilities can be treated as scalar quantities.
3. Permeabilities are assumed to be functions of p_c only, i.e., insensitive to flux rates or other variables.

6.4.3. Scaling the Governing Equations for Flow of Two Fluids

Obvious explicit scale factors are:

$$\mathbf{k_o \equiv k; \quad p_o \equiv p_e .}$$

Here k refers to the maximum permeability to the wetting fluid. The wetting-fluid permeability is selected as a scale factor because the wetting fluid may sometimes alter pore geometry, especially in cases where media contain swelling clays. Other scale factors are temporarily designated as:

$$\mathbf{x_o, \ t_o, \ q_o .}$$

Scaling Darcy's equation for the wetting fluid gives:

$$\mathbf{q_o \hat{q}_{wi} = -\frac{k\hat{k}_w}{\mu_w}\left[\frac{p_e}{x_o}\frac{\partial \hat{p}_w}{\partial \hat{x}_i} + \gamma_w \frac{\partial \hat{h}}{\partial \hat{x}_i}\right],} \qquad 6.9$$

where the hats indicate scaled variables. Constant coefficients inside the bracket on the right of the scaled equation can be set equal to unity only by defining

$$\mathbf{x_o \equiv \frac{p_e}{\gamma_w} .}$$

The remaining constant coefficient is set equal to unity by defining

$$\mathbf{q_o \equiv \left[\frac{k\gamma_w}{\mu_w}\right].}$$

The scaled equation for wetting fluid flux is

$$\mathbf{\hat{q}_{wi} = -\hat{k}_w \left[\frac{\partial \hat{p}_w}{\partial \hat{x}_i} + \frac{\partial \hat{h}}{\partial \hat{x}_i}\right] .} \qquad 6.10$$

The scaled equation for the nonwetting fluid with the same scale factors is

$$\mathbf{\frac{\mu_{nw}}{\mu_w}\hat{q}_{nwi} = -\hat{k}_{nw}\left[\frac{\partial \hat{p}_{nw}}{\partial \hat{x}_i} + \frac{\gamma_{nw}}{\gamma_w}\frac{\partial \hat{h}}{\partial \hat{x}_i}\right].} \qquad 6.11$$

Equation 6.11 contains two constant coefficients, the ratios of viscosities and specific weights, that cannot be set equal to unity with the scale factors used. Scale factors could be selected to eliminate the coefficients in this equation, but in that case they would appear in the wetting fluid flux equation.

All constant coefficients in the material balance equation can be set equal to unity by defining

$$t_o \equiv \left[\frac{\phi\,\mu_w\,p_e}{k\,\gamma_w^2}\right].$$

The scaled versions of the material balance equations are

$$\frac{\partial S}{\partial \hat{t}} = -\frac{\partial \hat{q}_{wi}}{\partial \hat{x}_i} = \frac{\partial \hat{q}_{nwi}}{\partial \hat{x}_i}. \qquad 6.12$$

Other scaled equations in the set are

$$\hat{k}_w,\ \hat{k}_{nw},\ S = f(\hat{p}_c), \qquad 6.13$$

and

$$\frac{\partial \hat{p}_c}{\partial \hat{x}_i} = \frac{\partial \hat{p}_{nw}}{\partial \hat{x}_i} - \frac{\partial \hat{p}_w}{\partial \hat{x}_i}. \qquad 6.14$$

6.4.4. Criteria of Similarity for Flow of Two Fluids

An inspection of the set of scaled equations indicates that requirements for similarity are:

1. geometric similarity, same scaled size, and same orientation,
2. same initial and boundary conditions in terms of scaled variables,
3. equal viscosity and density ratios for the two fluids occupying the media,
4. same relationships among scaled values of capillary pressure, saturation, and permeabilities.

An additional criterion of similarity may be required for cases in which the nonwetting fluid is a gas, e.g., water-air systems in fine-textured soils, where gas slippage might be significant. In the latter case, equal ratios of maximum water permeability to maximum air permeability are also required.

6.4.5. Comparison of Scale Factors for 2-phase Flow

Investigators, including both soil and petroleum scientists, have presented scale factors more or less equivalent to those presented here. However, soil scientists usually do not consider equality of viscosity ratios as a requirement for similarity because they believe this ratio to be negligible for air-water systems. Because resistance to flow of air is assumed to be negligible, k_{nw} is not one of the variables considered by soil scientists.

Earlier investigators of similarity in the petroleum industry often included a criterion evaluating inertial effects. Later investigators [Rapoport (1955), and Loomis and Crowell (1964)] in the petroleum field noted that this is usually not necessary. Miller and Miller (1956) included a requirement that porosities of media be equal, because they used volumetric water content θ, rather than saturation S for characterizing fluid contents.

Although authors usually agree in respect to explicit statements of similarity requirements, their criteria are in reference to different sets of scaled variables. An examination of scale factors employed by various authors is needed for an adequate comparison of proposed similarity criteria. Furthermore, the usefulness of a particular scheme depends upon how easily and precisely scale factors can be identified and evaluated, as well as upon how precisely the relationship among variables coalesce. Table 6.1 compares scale factors used by a petroleum scientist with those proposed by Brooks and Corey (1966) and Miller and Miller (1956). The symbols used in the original papers have been replaced (where possible) by corresponding symbols used elsewhere in this text.

The symbol L refers to a macroscopic length which characterizes the size of the entire flow system, $f(\alpha)$ is a function of the contact angle of the interfaces and $\Delta\gamma$ designates $\Delta\rho g$.

Table 6.1. Scale factors of three authors

Unit	Richardson (1961)	Miller and Miller (1956)	Brooks and Corey (1966)
Length	$\frac{\sigma f(\alpha)}{\Delta\gamma} / \sqrt{k/\phi}$	$\sigma / \gamma_\omega d$	$p_d / \Delta\gamma$
Capillary pressure	$\sigma f(\alpha) / \sqrt{k/\phi}$	σ / d	p_d
Time	$\frac{\mu L \phi}{k \Delta\gamma}$	$\frac{\mu L}{\gamma_\omega d^2}$	$\frac{\mu p_d \phi_e}{k(\Delta\gamma)^2}$
Flux rate	$\frac{k \Delta\gamma}{\mu}$	$\frac{\gamma_\omega d^2}{\mu}$	$\frac{k \Delta\gamma}{\mu}$

The scale factor for capillary pressure used by Brooks and Corey (1966) is p_d rather than the entry pressure p_e. The two scale factors often have values that are close but not identical, p_e being slightly larger than p_d. Entry pressure is defined as the capillary pressure at which the nonwetting fluid first becomes continuous during a desaturation process, whereas p_d is found by an extrapolation process as explained in Section 2.3.6.

Scale factors shown in Table 6.1 for macroscopic length are, in every case, equal to the scale factor for capillary pressure divided by the difference in specific weights. Furthermore, the scale factors are inversely proportional to a characteristic pore dimension. The characteristic pore dimension is represented by $\Delta\rho g / p_d$ in the scale factor of Brooks and Corey (1966) and by a function of $\sqrt{k/\phi}$ in the petroleum literature. Only Miller and Miller (1956) have included the characteristic pore dimension d explicitly.

The scale factor for length presented in this chapter is different from the Brooks-Corey factor in that capillary pressure is divided by the specific weight of wetting fluid rather than the difference in specific weights. This is not a significant difference since the ratios of specific weights must be equal for similarity, so the specific weight of wetting fluid differs from the difference in specific weights by a common multiplier.

According to theory presented in Section 2.4 leading to Equation 2.13, if d is assumed to be proportional to a maximum value of A/wp, it is also proportional to $\sigma \cos\alpha / p_d$. If α is a constant, σ / d is proportional to p_d. Furthermore, according to the theory presented in Section 3.6.1, p_d is inversely proportional to $\sqrt{k/\phi}$, provided the parameters σ, α, T, k_s and S_r are approximately equal. To the extent this theory is valid, all the scale factors presented in Table 6.1 for both capillary pressure and length are equally valid because they are mutually proportional. However, p_d should be an appropriate scale factor for p_c regardless of the validity of the theory involving the microscopic variables.

Entry pressure p_e is suggested here as an alternative to p_d as a scale factor for capillary pressure. Any capillary pressure characterizing the scale of capillary pressure-saturation functions is suitable for this purpose, provided its value can be determined with adequate precision.

The scale factor for macroscopic length should be inversely proportional to the entry pressure. The reason becomes clear if one considers a simple 1–dimensional static case where two fluids (say air and water) exist as continuous phases in a column of soil with a water table ($p_c = 0$) at the bottom of the column. The soil column, remains nearly fully saturated with water above the water table for a height that depends on the magnitude of $p_e / \Delta\gamma$. Clearly, if two such columns are to have similar distributions of water content along their length, the ratio of overall length L of the column to $p_e / \Delta\gamma$ must be the same.

Although p_e is inversely related to the size of the largest pores that form a continuous channel within the matrix, the relationship depends upon some variables not easily measured as they exist within the pore space, e.g., σ, α, and A/wp. Consequently, the entry pressure seems to be a more appropriate scale factor for length since it directly characterizes the scale of the capillary pressure-saturation function.

Leverett et al. (1942) were the first to employ the parameter $\sigma / \sqrt{k/\phi}$ as a scale factor for capillary pressure. This results in

$$\mathbf{J(S) \equiv \frac{p_c}{\sigma} \sqrt{k/\phi}}$$

being one of the variables often used to describe flow in petroleum reservoirs. The variable J(S) is often called the "J" function in petroleum literature.

The only difference between the scale factor J(S) and that used by Richardson (1961) is the contact angle introduced by the latter. Perkins and

Collins (1960), whose scale factors are not shown in Table 6.1, proposed essentially the same set of scale factors except that they replaced k with k_m, permeability with entrapped nonwetting fluid.

Scale factors for time, with the exception of that of Miller and Miller (1956), are essentially the same. The time scale factor of Brooks and Corey (1966) includes $p_d / \Delta\gamma$ rather than L but this is not a significant difference, since their theory requires L to be proportional to $p_d / \Delta\gamma$ for similar systems. Miller and Miller (1956) are the only authors in this group who have not included porosity in the time scale factor, since they have used the volumetric water content θ, rather than the saturation S, as a measure of wetting fluid content.

The scale factors for flux rate shown in Table 6.1 also are identical, except for that of Miller and Miller (1956) who have introduced the parameter d in this scale factor rather than the permeability k. The validity of the use of d in the scale factors for both flux rate and time depends upon the validity of the theory presented in Section 3.6.1. The scale factors for flux rate and time in terms of k and ϕ should be appropriate in any case. Earlier investigators in the petroleum industry [Leverett et al. (1942)], like Miller and Miller, considered similar pore geometry to be necessary for similar flow behavior.

6.4.6. Similarity for Special Cases

Requirements for similarity in the most general case are very difficult to satisfy. However, not all of the governing equations for flow of two fluids apply in every case, and some of the criteria listed for the general case are not always required. Rather than guessing which criteria to drop from the more general set, it is usually safer to write the simplest equations that adequately describe flow for the special case and derive similarity criteria specific for the system under consideration.

A special case involving flow of a wetting liquid only was described in Section 6.3.3. In that case, one might have guessed that the last two criteria on the list could be dropped and the others retained. However, the best way of stating criteria of similarity for a special case may not be obvious a priori.

Another example of how criteria of similarity can be simplified for special cases has been presented by Brooks and Corey (1966). The cases for which their simplification applies are those for which the saturation of the wetting fluid is never reduced below residual saturation. This condition is satisfied where the wetting fluid is removed by fluid flow only and not by diffusion or evaporation. In this case there is typically a portion of the pore space always occupied by adsorbed (or other practically immobile fluid) which contributes negligibly to permeability of the media.

Where the Brooks-Corey simplification applies, it is possible to normalize the porosity appearing in the material balance equation as

$$\phi_e \equiv \phi - \phi_r .$$

The parameter ϕ_e is called "effective porosity," and ϕ_r is called the "residual porosity," which represents a portion of the pore space contributing negligibly to permeability. The variable S is likewise normalized as

$$S_e \equiv \frac{S - S_r}{1 - S_r}$$

and is called "effective saturation."

The left side of the material balance equation is expressed in terms of the normalized quantities as

$$\phi_e \frac{\partial S_e}{\partial t} .$$

The latter substitution represents the same quantity expressed in different variables, but the equation is otherwise unchanged.

Much of the variability characteristic of functional relationships among the scaled variables presented in Section 6.4.3 is removed by the substitution. Moreover, Brooks and Corey found they could express functional relationships among the normalized variables in a satisfactory manner by empirical and semi-empirical relationships as follows:

$$S_e = \hat{p}_c^{-\lambda}; \; \hat{k}_w = \hat{p}_c^{-(2+3\lambda)}, \qquad 6.15$$

and

$$\hat{k}_{nw} = \left[1 - \hat{p}_c^{-\lambda}\right]^2 \left[1 - \hat{p}_c^{-(2+\lambda)}\right], \qquad 6.16$$

where $\hat{p}_C \geq 1.0$.

$$\mathbf{S},\ \hat{\mathbf{k}}_w,\ \text{and}\ \hat{\mathbf{k}}_{nw} = \mathbf{1.0}\ , \tag{6.17}$$

where $\hat{p}_c \leq 1.0$.

As explained in Section 2.3.6, λ is a positive exponent with a typical value of about 2 for porous rocks. It usually has a value less than 2 for structured soils and greater than 2 for clean sands.

To the extent that the Brooks-Corey relationships are valid, the requirement for identical relationships among scaled variables of capillary pressure, saturation, and permeabilities is satisfied by media having equal values of λ. As explained in Section 2.3.6, the value of this exponent can be found from the slope of a log-log plot of a capillary pressure-saturation curve. It can be found more conveniently by determining a best-fit exponent directly from desaturation data using a computer. When determining values of the exponent, data for wetting fluid saturations greater than about 0.85 should be excluded. This is because the Brooks-Corey theory does not apply where the nonwetting fluid is discontinuous.

Brooks and Corey originally expected their simplified criterion of similarity to be valid only for drainage of soils where hysteresis does not occur. Corey and Corey (1967), and McWhorter and Corey (1967) verified the Brooks-Corey theory experimentally for 1-dimensional drainage of soil columns. It was verified for 2-dimensional drainage by Hedstrom et al. (1971). However, the problems of identifying similarity requirements for media undergoing hysteresis, non-isotropic media, and media undergoing shrinking or swelling, need more research.

Linear imbibition of a liquid into a horizontal tube of homogeneous soil at some initial liquid content (see Section 5.2.1) is an example of a case where geometric similarity with respect to three dimensions is not necessary. The source of the liquid is at one end of the tube and the pressure of the liquid at the source is equal to or slightly greater than atmospheric. The other end of the tube is open to the atmosphere so resistance to the escape of air ahead of the advancing liquid is expected to be negligible.

In choosing a specific equation to describe this case, it is important to note that the source pressure is such that a portion of the tube of soil is fully saturated after a finite interval of time following the start of the imbibition process. Equation 5.5 does not apply in the fully saturated region since θ is not a single-valued function of ψ. The appropriate equations are simplified versions of Equations 5.1 - 5.4.

Another case for which a two-fluid flow system is often idealized as a linear flow problem occurs when brine is injected from a line of wells at a constant rate and fluids are removed from a line of wells at a distance L from the injection wells. It is assumed that the initial saturation of brine in the rock formation is uniform and is equal to or greater than the residual saturation. The displacement takes place under a very large pressure gradient and the slope of the formation is very small. Equation 5.41 is an appropriate equation for this case, but since the displacement process does not violate the conditions under which the Brooks-Corey simplifications are valid, the variables S and ϕ can be normalized. As explained in Section 5.4.1, the equation can be further simplified for this particular case by writing it in the form of the Buckley-Leverett equation.

Evidently, a number of specific equations involving different sets of variables may have to be tried to find the least restrictive set of similarity requirements. Furthermore, the least restrictive set may not always be apparent from an inspection of the set applicable to the most general case of 2-phase flow.

6.4.7. Tests of Similarity Criteria

Several experiments have been conducted by soil scientists to test the similarity theory of Miller and Miller (1956). These include experiments by Klute and Wilkinson (1958), Elrick et al. (1959) and Wilkinson and Klute (1959). The tests compared results (in terms of transformed variables) of flow experiments with two or more media having contrasting permeabilities and textures but satisfying the criteria for "similar" media. In one case, Elrick et al. (1959) ran hysteresis loops of $p_c(S)$ functions using two different fluids (water and butyl alcohol) on the same media. The agreement was good when an inert medium was used but not as good with a medium containing clay. The varying reaction of the clay to the two fluids undoubtedly prevented the retention functions from behaving in a similar way with each of the fluids. In other cases, however, tests have shown the theory of Miller and Miller (1956) to be valid.

Corey and Corey (1967) reported tests of the Brooks-Corey similarity theory for linear drainage of initially fully saturated soil columns with a zero p_c maintained at the base of the columns. Outflow as a function of time was recorded in terms of scaled as well as standard variables. To obtain maximum precision and reproducibility of measured parameters, Soltrol was used instead of water as the wetting fluid. The lengths of the columns were adjusted to be the same multiple of p_d / γ for each of the two materials having the same value of λ. The results are shown in Figures 6.1 and 6.2.

Hedstrom et al. (1971) conducted 2-dimensional drainage experiments in two soil flumes, the larger flume being 12.2 meters long and 1.22 meters high and the smaller flume having dimensions about 1/3 as large. Soltrol was used rather than water in this experiment to improve the stability of the soil structure and to permit maximum precision in measurement of the soil parameters. Examples of data obtained for discharge as a function of time in terms of standard units and scaled variables are shown in Figures 6.3 and 6.4, respectively. Data obtained for the elevation, of the water table as a function of time also coalesce very closely. From such experiments, Hedstrom et al. (1971) concluded that the Brooks-Corey similarity criterion is probably valid for 2-dimensional drainage, and that results from the small flume could be extrapolated to apply to larger systems.

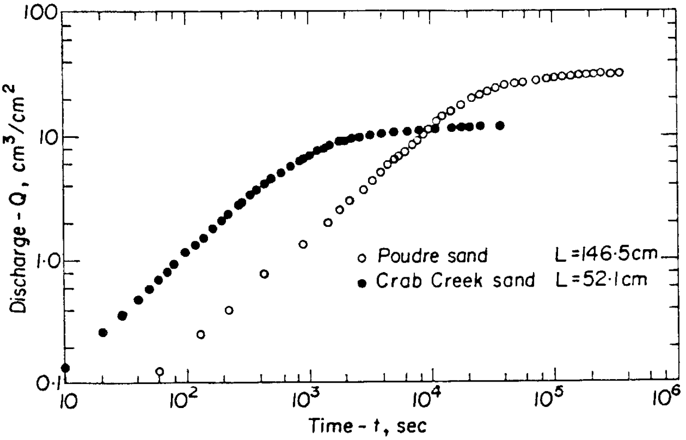

Figure 6.1. Outflow as a function of time (in terms of standard units) from columns of equal scaled height containing media of identical λ. [After Corey and Corey (1967)].

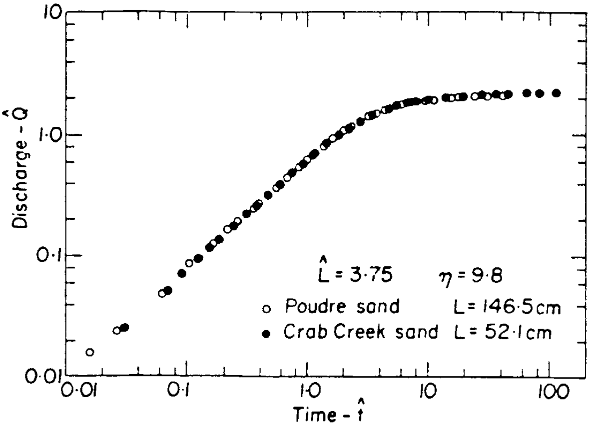

Figure 6.2. Outflow as a function of time (in terms of scaled variables) from columns of equal scaled height containing media of identical λ. [After Corey and Corey (1967)].

The experiments of Corey and Corey (1967), and Hedstrom et al. (1971) represent cases in which the resistance to flow of a nonwetting phase is undoubtedly negligible. McWhorter and Corey (1967) reported results of a test for a case where resistance to flow of nonwetting fluid (air) is significant. Details of this experiment have been presented in a thesis by McWhorter (1966), a brief description of which is given below.

Three columns of porous media were first fully saturated with Soltrol and then drained under the influence of gravity for a period corresponding to a scaled time of 1.0. At that time, a scaled air pressure of 1.2 was applied at the top of each column. The cumulative outflow from each column was measured as a function of time for the entire period. One of the columns was a core of Berea sandstone, 114 cm in length. The other two were shorter columns of unconsolidated sand, all of them, however, having the same height relative to respective values of $p_d / \Delta\gamma$. The sandstone core was 3.6 times as long as the shorter of the two unconsolidated sand columns (in terms of standard units) and required 108 times as long to drain to an equivalent saturation.

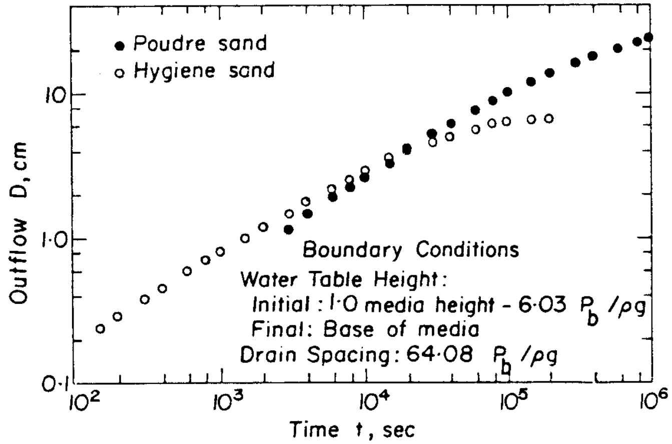

Figure 6.3. **Cumulative outflow from 2-dimensional drainage models with similar media. [Hedstrom et al. (1971)].**

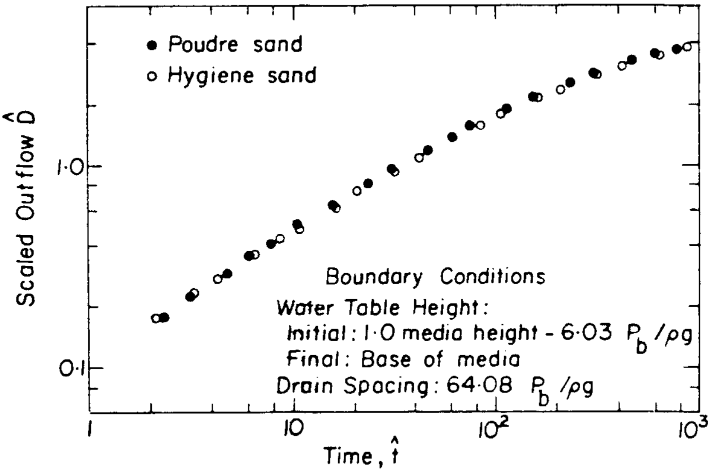

Figure 6.4. **Cumulative outflow from 2-dimensional drainage models with similar media (in terms of transformed variables). [Hedstrom et al. (1971)].**

The experimental results are presented in Figures 6.5 and 6.6. A sharp increase in liquid discharge occurred at a scaled time of 1.0, as would be expected, because this is when an elevated air pressure was applied. Although the data in terms of standard units are very different for the three columns, they coalesce closely when expressed in terms of scaled variables.

The three media in the McWhorter experiment had λ values that were not greatly different, being about 2.6 for two of the columns and 2.1 for the third. Evidently, 1-dimensional drainage of sands is not extremely sensitive to small differences in λ. A similar observation was made by Corey and Corey (1967) for media with large values of λ, i.e., greater than 2.0.

The columns did not satisfy criteria for "similar media" as described by Miller and Miller (1956) because the porosity of the sandstone was less than half that of the unconsolidated sands. Moreover, the ratio of length of the sandstone column to that of the volcanic sand was 3.6 rather than 5.86 as would have been required using the Leverett length scale. Evidently, the Brooks-Corey normalized variables adequately account for porosity differences, and the Brooks-Corey length scale seems to be more satisfactory for scaling dimensions of porous media models than the Leverett length scale.

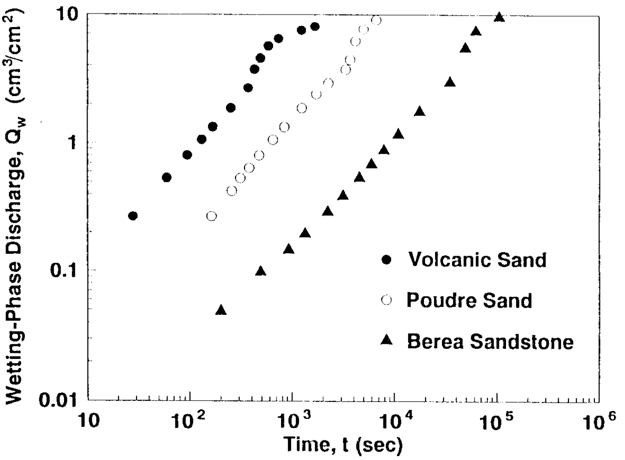

Figure 6.5. Discharge as a function of time for the wetting phase driven by air pressure for three columns of porous media with nearly equal values of λ.

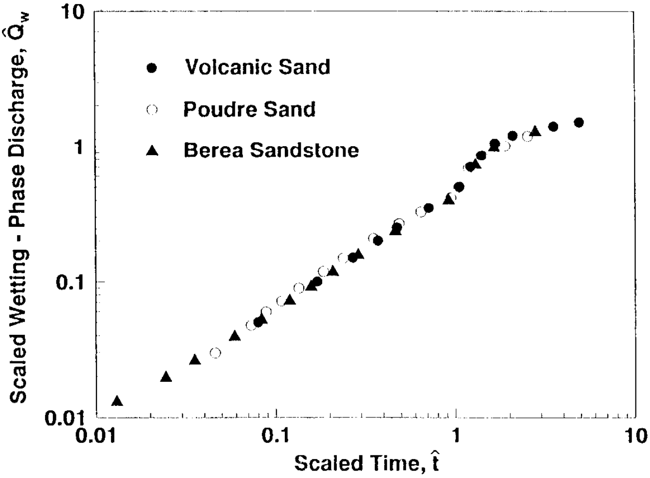

Figure 6.6. Discharge as a function of time (in terms of scaled variables) for the wetting phase driven by air pressure for three columns of porous media with nearly equal values of λ.

PROBLEMS AND STUDY QUESTIONS

1. For model studies involving flow in open channels, the Froude number is an important criterion of similarity. This number includes a characteristic length L. What is the only logical dimension to use for this purpose? Explain. Is it necessary to use this dimension in a Froude number if similarity between two systems with complete geometric similarity is under consideration? Explain.

2. Consider the problem of building a model to study the effect of soil properties on distribution of soil water under furrows used in irrigation. Would complete geometric similarity be necessary? Explain.

3. Explain why the criteria of similarity used by Brooks and Corey for drainage situations would have to be modified for imbibition into dry soil.

4. Consider the distribution of water in a homogeneous soil profile that has been drained to a static condition with a water table at some depth. Describe a simple laboratory system that would be effective as a model for the prototype system. Develop similarity criteria for the variables h and S from the equation of statics and the Brooks-Corey relationship for $S(P_c)$. Show what criteria could be eliminated if the variables are $\hat{h}$ and S_e.

5. Consider a case of 1-dimensional steady flow of liquid through small diameter horizontal tubes of soil that contain a static interconnected gas phase. Derive the least restrictive similarity criteria for a set of variables that will describe flux rate and the capillary pressure distribution.

6. Derive the least restrictive similarity {for tubes of soil as described in problem 5} if the wetting phase is static and the gas is flowing.

7. What is the purpose of ϕ in the scale factor for p_c used by petroleum scientists? Does ϕ in the J-function, make the similarity criteria less restrictive if it is required that ϕ be equal? Does it make the criteria of Richardson less restrictive? Explain.

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